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Understanding the Concept of Limit in Calculus

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ABSTRACT : Calculus is a highly important foundational mathematics course offered in the first semester for most science and engineering majors in universities across the country. Among all mathematics courses, it has the largest total number of class hours, which underscores its significance. In the calculus course, the most crucial concept is the limit. When restricted to use just one word to summarize all the contents in calculus, "limit" would be the definitive choice. Despite its importance, the concept of limit is hard to grasp. This paper aims to provide a detailed explanation of the concept of limit to facilitate a comprehensive understanding and mastery of this critical topic.

KEYWORDS - mathematical analysis, calculus, advanced mathematics, limit.

I. INTRODUCTION

Calculus is a landmark achievement in the history of human science, developed independently by Newton and Leibniz in the 17th century, each contributing uniquely to its foundation. In contemporary society, any college or university in any country worldwide offers calculus-related courses. Thus, calculus is a highly important basic course. It is not an exaggeration to say that the mastery degree of calculus almost directly affects the learning of subsequent mathematics courses. Limit is a very important term in calculus which is introduced at the very beginning of the course and permeates all chapters of the subject. Therefore, the understanding and mastery of the concept of the limit directly influence the learning of calculus as a whole. In other words, to excel in calculus, one must first comprehend and master the concept of the limit.

At first glance, the concept of limit may seem simple, but in fact, it is the core and the most difficult concept to understand in calculus. While its intuitive understanding appears straightforward, its mathematical definition is much harder to grasp. Why is a mathematical definition of the limit necessary? The answer lies in its utility for rigorous theoretical proofs, which natural language definitions cannot provide. At the beginning of the establishment of calculus, for over a century, many mathematicians put forward their interpretations of the limit, but none of them could eliminate the intuitive traces in the concept of limit. As a result, a rigorous mathematical definition of the limit could not be given. Nowadays, the mathematical definition of limit that is widely accepted and written into calculus textbooks was introduced by Karl Weierstrass. It was precisely the proposal of this definition that provided calculus with a solid theoretical foundation.

Today, the limit is not merely a concept, but also a profound ideological approach for human beings to solve numerous difficult problems, with its influence seen in many domains. Thus, it is essential for instructors to provide an in-depth explanation of the mathematical definition of the limit in class so that students can achieve a thorough understanding and mastery of it. However, quite a number of university mathematics instructors explain the definition of the limit from the perspective of trends, which still fail to deviate from the intuitive traces in the concept essentially. This will lead to students' insufficient understanding of the mathematical definition of the limit or leave them stuck at the stage of intuitive understanding, thus posing challenges for the later explanations of proving the properties of the limit with the definition. When Weierstrass formulated the mathematical definition of the limit, he approached it from a static perspective, abstracting the natural language definition to create a universally recognized mathematical definition. To ensure students fully grasp the mathematical definition of limit, besides explaining it from the perspective of the dynamic trend, more attention should be paid to illustrating from the static and abstract perspective. This approach makes the definition easier to understand, facilitates the comprehension and proof of limit properties, and helps students apply the definition to calculate the limits of specific sequences and functions. Ultimately, this approach greatly benefits students' learning of calculus as a whole. This paper provides a detailed explanation of the definition of the limit, especially the transition from the intuitive understanding to the static and abstract mathematical definition. The discussion is organized as follows: it first explains the mathematical definition of the limit of a sequence, followed by that of a function.

Since a sequence is a special type of function, understanding the mathematical definition of the limit of a sequence facilitates the comprehension of the mathematical definition of the limit of a function.

II. THE LIMIT OF A SEQUENCE

In the real world, "limit" is a very common term. Its meaning may vary slightly depending on the context. For sequences or single-variable functions, the limit is nothing but a real number. This is crucial and should be clearly pointed out to students straightforwardly in class at first. Suppose $\{a_n\}$ represents any sequence, and A represents a real number. Intuitively, the limit of a sequence $\{a_n\}$ being A means that $\{a_n\}$ will approach A infinitely. That is, when the index n is sufficiently large, a_n will approach A infinitely. This indicates that when n is sufficiently large, the distance $|a_n - A|$ between a_n and A will approach zero infinitely. Now, please think about how to represent that n is sufficiently large and that the distance $|a_n - A|$ will approach zero infinitely. If this problem seems difficult to grasp immediately, let us consider another question: how can one draw a giant on an A4 sheet of paper with a length of 297 millimeters and a width of 210 millimeters. With a moment's thought, it becomes clear: one could start by drawing the legs of a person on the A4 paper, then add a very small figure beside these legs, and finally inform the reader that this small person represents Yao Ming. Upon further consideration, with Yao Ming positioned at the feet of this figure, would this figure not be considered as a giant?

In fact, in real life, it is common to use references to illustrate problems. If you want to show that you are tall, you just need to find a very tall person and then tell others that you are taller than him. If you want to show that you are handsome, you can simply claim that you are more handsome than a certain handsome guy. If you want to show that your girlfriend is beautiful, you only need to... Upon careful consideration, we realize that this reasoning is valid because attributes such as size, quantity, height, thinness, or beauty are comparative terms. Only through comparison can results be drawn. Returning to our original problem: How can we express that the distance $|a_n - A|$ approaches zero infinitely? Since $|a_n - A|$ is greater than or equal to zero, this problem can be transformed into expressing that $|a_n - A|$ is infinitely small. When we see "small", we realize it can be expressed through comparison. To achieve this, we can use an arbitrarily small positive number $\epsilon > 0$. If the distance $|a_n - A|$ between a_n and A is smaller than ϵ, it implies that for any $\epsilon > 0$, there exists $|a_n - A| < \epsilon$, which shows that $|a_n - A|$ between a_n and A approach zero infinitely. Similarly, to express that the number of terms n is sufficiently large, we need to find a particularly large positive integer N. If n is larger than the particularly large positive integer, it means that n is sufficiently large. It is important to emphasize that once such a positive integer N is identified, it always exists. Therefore, "there exists a positive integer $N > 0$, $n > N$" can express that n is sufficiently large. Based on this, the mathematical definition of the limit of a sequence can be described as follows:

Definition 1. Let $\{a_n\}$ be a sequence and A be a fixed constant. If for any given positive number ε, there exists a positive integer N such that when $n > N$, $|a_n - A| < \varepsilon$ holds, then we say that the limit of the sequence $\{a_n\}$ is A, or the sequence $\{a_n\}$ converges to A, and we write $\lim_{n\to\infty} a_n = A$ or $a_n \to A$ $(n \to \infty)$.

To help students better understand and grasp the above mathematical definition of the limit of a sequence, the following three points need to be explained:

a. The arbitrariness of ϵ: The role of the positive number ϵ in the above definition is to measure how close the general term a_n of the sequence are to the constant A. The smaller ϵ is, the closer a_n is to A. The positive number ϵ can be arbitrarily small, indicating that the general term a_n and the fixed constant A can be close to any extent. However, although ϵ is arbitrary, once it is specified, it is fixed temporarily, meaning that the arbitrarily given positive number ϵ does not depend on any value.

b. N emphasizes existence: Generally speaking, the positive integer N increases as the given positive number ϵ decreases. Therefore, in some calculus textbooks, N is often written as $N(\epsilon)$ to indicate that N depends on ϵ. In the definition above, the most important aspect of the positive integer N is emphasizing its existence. Details such as how many such integers N exist or how large each N is are generally not of concern.

c. Geometric meaning of the limit of a sequence: From the above definition, it is known that when $n > N$, $|a_n - A| < \epsilon$ holds. This means that all terms a_n with subscripts greater than the positive integer N lie within the neighborhood of A with radius ϵ, denoted as $U(A, \epsilon)$. Outside this neighborhood $U(A, \epsilon)$, there are at most N terms in the sequence $\{a_n\}$, that is, at most a finite number. Thus, an equivalent definition of the limit of a sequence can be given as follows:

The limit of the sequence $\{a_n\}$ is A if and only if for any given $\epsilon > 0$, almost all terms in the sequence $\{a_n\}$ lie within $U(A, \epsilon)$, and at most a finite number of terms lie outside $U(A, \epsilon)$.

The equivalent definition of the limit of a sequence above is derived from geometric intuition. Therefore, it can be viewed as the geometric meaning of the limit of a sequence. It is worth noting that the geometric meaning of the limit is very helpful for future problem-solving and understanding of their properties.

Example: In the following three statements, the correct number is ( ).

① If the sequence $\{a_n\}$ converges to A, then any subsequence $\{a_{n_i}\}$ also converges to A.

② If a certain subsequence $\{a_{n_i}\}$ of the sequence $\{a_n\}$ converges to A, then the sequence $\{a_n\}$ also converges to A.

③ If a certain subsequence $\{a_{n_i}\}$ of the monotonic sequence $\{a_n\}$ converges to A, then the sequence $\{a_n\}$ also converges to A.

④ If any subsequence $\{a_{n_i}\}$ of the sequence $\{a_n\}$ converges to A, then the sequence $\{a_n\}$ also converges to A.

⑤ If both the sequence $\{a_{2n}\}$ and $\{a_{2n-1}\}$ converge to A, then the sequence $\{a_n\}$ converges to A.

Options: (A) 1, (B) 2, (C) 3, (D) 4.

Solution: The correct answer is (D). If a sequence $\{a_n\}$ converges to A, then for any given $\epsilon > 0$, almost all terms of the sequence $\{a_n\}$ lie within $U(A, \epsilon)$, with at most a finite number of terms lie outside $U(A, \epsilon)$, and vice versa. The converse is also true, so ① and ④ are correct. Similarly, ② is wrong and ⑤ is correct.

If a subsequence $\{a_{n_i}\}$ of the monotonic sequence $\{a_n\}$ converges to A, it can be deduced that almost all terms of the sequence $\{a_n\}$ will lie within the neighborhood $U(A, \epsilon)$, and at most only a finite number of terms will lie outside of $U(A, \epsilon)$. Consequently, the sequence $\{a_n\}$ must converge to A, i.e., statement ③ is correct.

III. THE LIMIT OF A FUNCTION

Since a sequence can be seen as a special type of function, this section mainly follows the mathematical definition of the limit of a sequence to explain the mathematical definition of the limit of a function. There are six types of limits of functions, which are $x \to +\infty$, $x \to -\infty$, $x \to \infty$, $x \to x_0$, $x \to x_0^+$ and $x \to x_0^-$. First, let's consider the case of $x \to +\infty$. Intuitively, the limit of $f(x)$ at $+\infty$ is A if and only if $f(x)$ will approach the constant A infinitely when the independent variable x tends to $+\infty$. Regarding $f(x)$ as a_n, the independent variable x is similar to the term number n. Therefore, the limit definition of the function $f(x)$ at $+\infty$ is the same as that of the sequence $\{a_n\}$. To express that the independent variable x tends to $+\infty$, we only need to find a particularly large positive number M. If x is larger than the particularly large positive number M, it means that the independent variable x tends to $+\infty$. Similarly, for any arbitrarily small positive number ε, if the distance $|f(x) - A|$ between $f(x)$ and A is smaller than any arbitrarily small ε, it indicates that the distance $|f(x) - A|$ between $f(x)$ and A will tend to zero infinitely. As a result, the formal mathematical definition of the limit of the function $f(x)$ at $+\infty$ can be described as follows:

Definition 2. Let the function $f(x)$ be defined in a certain neighborhood $U(+\infty)$ of $+\infty$, and A be a fixed constant. If for any given positive number ε, there exists a positive number M such that when $x > M$, $|f(x) - A| < \varepsilon$ holds, then we say that when the independent variable x tends to $+\infty$, the limit of the function $f(x)$ is A, and it is written as $\lim_{x\to+\infty} f(x) = A$ or $f(x) \to A$ $(x \to +\infty)$.

For the above - mentioned mathematical definition of the limit of a function, in order to enable students to have a more sufficient understanding and mastery, we also need to give the following two - point explanations:

a. The positive number M in Definition 2 plays the same role as the positive integer N in the definition of the limit of a sequence, which is used to measure the extent to which the independent variable x tends to $+\infty$. However, the difference is that the independent variable x here can take all real numbers larger than M, not just positive integers. Hence when $x \to +\infty$, the limit of the function $f(x)$ being A means that any small neighborhood of A must contain all function values of $f(x)$ near $+\infty$.

b. The geometric meaning of Definition 2: For any given positive number ε, there always exists a vertical line $x = M$ such that the curve $f(x)$ to the right of the line $x = M$ will all lie between the two horizontal lines $y = A + \varepsilon$ and $y = A - \varepsilon$ which are centered at A with a radius of ε and parallel to the x-axis (as shown in the Fig.1).

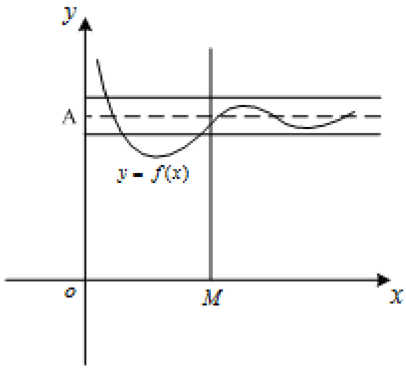

Fig.1 The geometric meaning of Definition 2

Similar to the definition of the limit of the function $f(x)$ at $+\infty$, the definitions of the limits of the function $f(x)$ at $-\infty$ and ∞ can be defined as follows:

Definition 3. Let the function $f(x)$ be defined in a certain neighborhood $U(-\infty)$ of $-\infty$, and A be a fixed constant. If for any given positive number ε, there exists a positive number M, such that when $x < -M$, $|f(x) - A| < \varepsilon$ holds, then we say that when the independent variable x tends to $-\infty$, the limit of the function $f(x)$ is A, and it is written as $\lim_{x \to -\infty} f(x) = A$ or $f(x) \to A$ $(x \to -\infty)$.

Definition 4. Let the function $f(x)$ be defined in a certain neighborhood $U(\infty)$ of ∞, and A be a fixed constant. If for any given positive number ε, there exists a positive number M, such that when $|x| > M$, $|f(x) - A| < \varepsilon$ holds, then we say that when the independent variable x tends to ∞, the limit of the function $f(x)$ is A, and it is written as $\lim_{x \to \infty} f(x) = A$ or $f(x) \to A$ $(x \to \infty)$.

Similarly, we can give the limits, left - hand limits and right - hand limits of $f(x)$ at the point x_0. Let the function $f(x)$ be defined in a certain punctured neighborhood $U^{\circ}(x_0)$ of the point x_0. To express that the independent variable x approaches x_0, we only need to find a particularly small positive number δ. If the distance $|x - x_0|$ between the independent variable x and x_0 is smaller than the particularly small positive number δ, it indicates that the independent variable x approaches x_0 infinitely. Similarly, for any arbitrarily small positive number ε, if the distance $|f(x) - A|$ between $f(x)$ and A is smaller than any arbitrarily small ε, it indicates that the distance $|f(x) - A|$ will approach zero infinitely. Therefore, the formal mathematical definition of the limit of the function $f(x)$ at x_0 can be described as follows:

Definition 5. Let the function $f(x)$ be defined in a certain punctured neighborhood $\mathring{U}(x_0)$ of the point x_0, and A be a fixed constant. If for any given positive number ε, there exists a positive number δ such that when $0 < |x - x_0| < \delta$, $|f(x) - A| < \varepsilon$ holds, then we say that when the independent variable x approaches x_0, the limit of the function $f(x)$ is A, and it is written as $\lim_{x \to x_0} f(x) = A$ or $f(x) \to A$ $(x \to x_0)$.

To ensure that students fully understand and grasp the mathematical definition of the limit of a function, it is also essential to provide the following two points of clarification:

a. The positive number δ in Definition 5 is used to measure the extent to which the independent variable x tends to x_0. However, it should be noted that the independent variable x here can take all real numbers in $\mathring{U}(x_0)$, and it does not matter whether x can take the value of x_0, because the limit is unrelated to the function value at the point x_0.

b. The geometric meaning of Definition 5: For any given positive number ε, there always exists a vertical strip centered at $x = x_0$ with a radius of δ, such that the curve $f(x)$ within this vertical strip fall between the two straight lines $y = A + \varepsilon$ and $y = A - \varepsilon$ which are centered at A with a radius of ε and parallel to the x-axis (as shown in Fig.2).

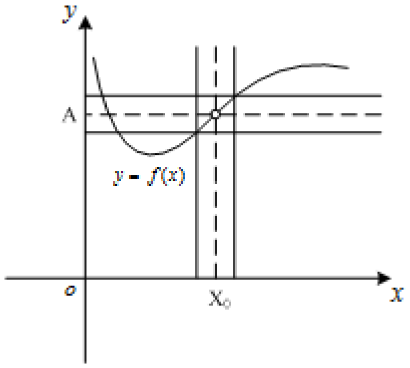

Fig.2 The geometric meaning of Definition 5

Similar to the definition of the limit of the function f(x) at x_0, we can define the left - hand limit and right - hand limit of the function f(x) at x_0.

Definition 6. Let the function f(x) be defined in a certain left - hand neighborhood $U^-(x_0)$ of the point x_0, and A be a fixed constant. If for any given positive number ε, there exists a positive number δ, such that when $x_0 - \delta < x < x_0$, $|f(x) - A| < \varepsilon$ holds, then we say that the constant A is the left - hand limit of the function f(x) at the point x_0, and it is written as $\lim_{x\to x_0^-} f(x) = A$ or $f(x) \to A\ (x \to x_0^-)$. Sometimes $f(x_0 - 0)$ is used to represent the left - hand limit value of f(x) at x_0.

Definition 7. Let the function f(x) be defined in a certain right - hand neighborhood $U^+(x_0)$ of the point x_0, and A be a fixed constant. If for any given positive number ε, there exists a positive number δ, such that when $x_0 < x < x_0 + \delta$, $|f(x) - A| < \varepsilon$ holds, then we say that the constant A is the right - hand limit of the function f(x) at the point x_0, and it is written as $\lim_{x\to x_0^+} f(x) = A$ or $f(x) \to A\ (x \to x_0^+)$. Sometimes $f(x_0 + 0)$ is used to represent the right - hand limit value of f(x) at x_0.

IV. CONCLUSION

The above are the mathematical definitions of the seven limits of sequences and functions. They all start from intuition, pass through static abstraction, and finally eliminate intuitive traces to obtain their formal mathematical definitions. Compared with the natural language definitions, the greatest advantage of the mathematical definitions of limits is that they can be applied to all theoretical proofs. A thorough understanding of the mathematical definitions of limits will lay a solid foundation for learning of calculus in future.

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