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An Approximation of the Unit Step Function: A New Method

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ABSTRACT : In the following paper, the author has derived an exact form of the Unit step function along with a better way to approximate the Heaviside step function which is a fundamental part of Operational Calculus. Particularly, in this paper, the function is expressed as a limit of a nested exponential function. The novelty of this paper is that it does not depend upon non-elementary special functions, such as the Dirac delta function or Gauss error function and an approximation with a high degree of precision can be obtained with relatively less computational effort. Thus, it may be advantageous to some degree while performing computational procedures that are inserted into Operational Calculus techniques.

KEYWORDS: Exponential Representation, Nested Exponents, Unit Step Function

I. INTRODUCTION

The Heaviside step function, named after the engineer and mathematician Oliver Heaviside, is a discontinuous single–valued function, whose value is zero for negative argument and one for positive argument, it is usually denoted by **H(t)** or **u(t)**, where **t** represents time [1]. This function is used to model and describe the abrupt change in a system at a specific point in time. This function was introduced by Oliver Heaviside, who was an important pioneer in the study of electronics and also made a remarkable contribution to the field of Operational Calculus [2].

It plays a vital role in differential equations, digital signal processing, and several other areas where accurately capturing step changes is crucial for understanding and predicting how systems behave. In the realm of control systems, it helps depict sudden shifts in inputs, making it easier to analyze how dynamic systems respond. When it comes to signal processing, it precisely describes signals that change in an instant. So, it's incredibly helpful for the necessary calculations involved in the creative and practical design aspects, especially from an engineering standpoint. Meanwhile, there exist many analytic approximations to the unit step function. Some of the literature about the approximation of the unit step function can be seen in the references [3] [4]. Sullivan et al [5] approximated the unit step function by means of a linear combination of exponential functions. Nevertheless, the prevailing characteristic among the aforementioned functions is the utilization of non-elementary special functions, such as the Logistic function, Hyper function, or Error function, to finalize the specified function. Additionally, a substantial portion of their algebraic formulations comprises generalized integrals or infinitesimal terms, thereby introducing intricacies into the associated computational procedures.

II. TOWARDS AN EXACT FORM OF THE UNIT STEP FUNCTION

The unit step function $u_a(x)$ we consider is defined by:

$$u_a(x) := \begin{cases} 1 \; if \; x > a \\ 0 \; if \; x < a \end{cases} \tag{1}$$

III. CLAIM

The unit step function $u_a(x)$ defined by 1 is given by:

$$u_a(x) = \lim_{c \to \infty} m^{-k^{\left(-c\frac{x-a}{|x-a|}\right)}} \tag{2}$$

for $m > 1$ and $k > 1$.

III. PROOF

We define a function $w_a(x)$ such that:

$$w_a(x) > 0 \text{ iff } x > a$$

$$\mathrm{w_a(x) < 0 \text{ iff } x < a}$$

For $\mathrm{k > 1}$, we define another function $\mathrm{v_a(x)}$ such that:

$$v_a(x) = \lim_{c\to\infty} k^{-cw_a(x)} \tag{3}$$

Here, $\mathrm{v_a(x)}$ approaches 0 when $\mathrm{x > a}$ and it approaches infinity when $\mathrm{x < a}$. Finally, $\mathrm{u_a(x)}$ is obtained by:

$$u_a(x) = m^{-v_a(x)} \tag{4}$$

The function $\mathrm{u_a(x)}$ as defined by (1) is achieved when $\mathrm{m > 1}$. Substituting 3 in 4 we get,

$$u_a(x) = m^{-\left(\lim_{c\to\infty} k^{(-cw_a(x))}\right)} \tag{5}$$

$$u_a(x) = \lim_{c\to\infty} m^{-k^{-(cw_a(x))}} \tag{6}$$

Also, $\mathrm{w_a(x)}$ can be realized by;

$$w_a(x) = \frac{x-a}{|\,x-a\,|} \tag{7}$$

Substituting 7 in 6 we get;

$$u_a(x) = \lim_{c\to\infty} m^{-k^{\left(-c\frac{x-a}{|x-a|}\right)}} \tag{8}$$

This completes the proof of the claim.

IV. DISCUSSION

The purpose of this study was to propose a new method for approximation of the unit step function. Particularly, this new method approximates the unit step function by taking the limit of a nested exponential function. The method's effectiveness lies in its simplicity and efficiency, as it achieves a remarkably high level of precision in approximation using comparatively small numerical values. A simple demonstration of its effectiveness is presented below;

E.g. Substituting $\mathrm{m = 7}$, $\mathrm{k = 10}$ and $\mathrm{c = 1}$ for $\mathrm{x < a}$ in 8, we get;

$$\mathrm{u_a(x) = 5^{-10^1}}$$

$$\mathrm{u_a(x) = 0.0000000035}$$

Additionally, another advantage of this particular method is that with a slight modification in the original equation, we are able to achieve the definition of $\mathrm{H_0(0)}$ as $1/2$ something that is taken for granted in the majority of the approximations of this function. For this, we represent $\mathrm{H_0(x)}$ as:

$$\mathrm{H_0(x) = \lim_{c\to\infty} m^{-k^{-cx}}}$$

Nonetheless, this approximation relies solely on algebraic representation, devoid of general integrals or special functions like the Dirac-delta function or the Error function. It may hold promise for computational procedures involving the application of the unit step function in operational calculations, as well as in engineering practice.

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V. CONCLUSION

This research introduces a new method for approximating the unit step function solely based on algebraic representation. Devoid of complex integrals and non-elementary special functions, the approach promises high precision and practical utility in operational calculations and engineering.

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BIOGRAPHIES AND PHOTOGRAPHS

Manish Adhikari, born on October 22, 2002, in the rural village of Nepal, is currently an undergraduate Electronics, Communication & Information Engineering student at Pulchowk Campus, Institute of Engineering, Tribhuvan University. The author has a profound interest in Mathematical Analysis and Discrete Mathematics as well as in the field of Artificial Intelligence. This research paper marks his venture into research on mathematical topics, reflecting his enthusiasm for exploring the intricate realms of mathematics. Beyond his academic pursuits, the author's achievements extend to being a national-level finalist in the Mathematics Olympiad. His journey reflects a dynamic blend of academic rigor and a passion for mathematical exploration, setting the stage for a promising future at the intersection of electronics and information engineering and mathematics.