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' Recurring Polygons Theorem: To study reoccurrence of polygons on drawing perpendicular lines to each side passing through vertices and find their dimensions.'

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ABSTRACT:
We all are familiar with polygons, In fact, we've been dealing with those since junior classes. Before I move on to the theory, I request you all to get ready with Pen, Paper, Scale, Compass, and Protractor and try this yourself. Follow the steps :

i. First, draw a hexagon (for convenience) of any side length you wish, say 'x' with the help of a compass and protractor.(*As you might have learned in junior classes; If not I would recommend you to learn it first*)

ii. Secondly, draw perpendicular lines to each side passing through vertices. You'll find that the lines intersect to form the same polygon but with different side lengths. (*As shown in **fig-b** of **Introduction***).

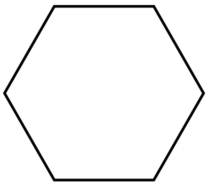
iii. Measure the length of the new polygon. Say it's 'y'. Repeat the same with the new hexagon. You'll get another hexagon of length say 'z'.

iv. Interestingly, x,y, and z forms a geometric progression where ;
$\frac{y}{x}=\frac{z}{y}=\frac{1}{\sqrt{3}}=cot(60°)=cot\left(\frac{120°}{2}\right)=cot\left(\frac{Interior\ Angle}{2}\right)$

v. It's not limited to hexagon. If you take any regular polygon and do so, the side length results in a GP with a common ratio cotangent of half of their interior angle.

I. INTRODUCTION:

Let us consider a regular polygon (say hexagon) as shown in fig (a). Let's draw lines perpendicular to each side passing through vertices. These lines overlap each other to give the same polygon but with different side length and twisted at 30 degrees as shown in fig (b) (*I' am not saying that only perpendicular lines would give the polygon, any line passing through vertices would overlap to give the same result but it's the special case I'm dealing with*) . If the same is repeated with the so formed hexagon and so on we get smaller and smaller hexagons as shown in fig (c), Let's call them daughter polygons.



Fig(a)

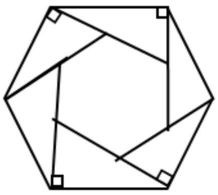
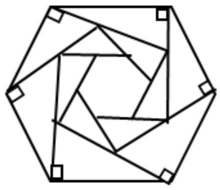
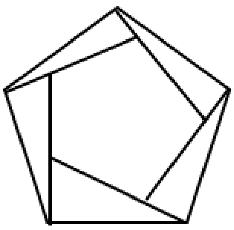


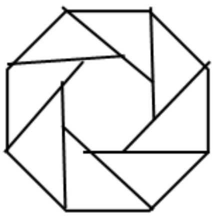
Fig (b)



Fig(c)

SIMILARLY, IF OTHER POLYGONS ARE TAKEN WE GET THE SAME RESULT;

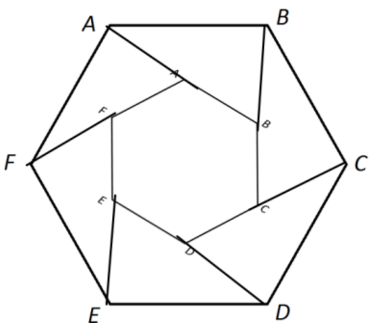


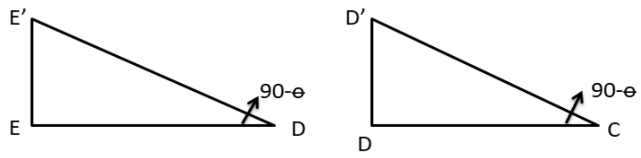


THEOREM:

For a given regular polygon of an interior angle θ and side length l, the side length of n^{th} formed daughter polygon is $l\left(cot\frac{\theta}{2}\right)^n$ and twisted to n$|\theta-90|$ with respect to the original polygon.

PROOF: For the proof let's take an arbitrary regular polygon ABCDEF of side length l and interior angle θ as shown in fig.





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Here, ABCDEF is the original polygon and A'B'C'D'E'F', the 1st daughter polygon. ED=l&E'D'=?
Using simple geometry and trigonometry we can determine the side E'D'. Here, We take two triangles EE'D & D'DC.(ΔEE'D≃ΔD'DC)

From ΔEE'D;

$cos(\theta-90)=\frac{l}{E'D}$

⇒E'D $=\frac{l}{sin\theta}$ -- (i)

From ΔD'DC;

$\Rightarrow tan(\theta-90)=\frac{D'D}{l}$

$\Rightarrow -\frac{cos\theta}{sin\theta}=\frac{D'D}{l}$

⇒ D'D = $-l\frac{cos\theta}{sin\theta}$ - - - - ----------------------------------- (ii)

Now,

E'D' = E'D-D'D

$= l\frac{1}{sin\theta}-\left(-l\frac{cos\theta}{sin\theta}\right)$ - - - - - - - - - - - - - - - - - { From (i) & (ii)}

$= l\left(\frac{1}{sin\theta}+\frac{cos\theta}{sin\theta}\right)$

$= l\left(\frac{1+cos\theta}{sin\theta}\right)$

$= l\left(\frac{2cos^2\frac{\theta}{2}}{2sin\frac{\theta}{2}*cos\frac{\theta}{2}}\right)$

$= \quad lcot\frac{\theta}{2}$

Here, the Side length of 1st daughter polygon = $lcot\frac{\theta}{2}$ i.e., $\cot\frac{\theta}{2}$ times previous and twisted to $\theta - 90°$

If the same is repeated with the 1st daughter polygon, we get another polygon whose side length would be $\cot\frac{\theta}{2}$ times the 1^{st} one and twisted to 2*$|\theta - 90|$w.r.t original polygon

i.e. Side length of 2nd daughter polygon $= \left(lcot\frac{\theta}{2}\right) * cot\frac{\theta}{2}$

" " " " $= l\left(cot\frac{\theta}{2}\right)^2$; twist angle=$2* |\theta - 90|$

" " 3^{rd} " $= l\left(cot\frac{\theta}{2}\right)^3$; twist angle=$3* |\theta - 90|$

⇒ The side length of n^{th} daughter polygon = $l\left(cot\frac{\theta}{2}\right)^n$; twist angle=$n* |\theta - 90|$

II. RESULT:

It turns out that the side length of successive polygons results in a geometric progression with a common ratio$cot\frac{\theta}{2}$.

SPECIAL CASES AND ILLUSTRATIONS:

Earlier we saw polygons getting smaller and smaller, but it's not always the case.

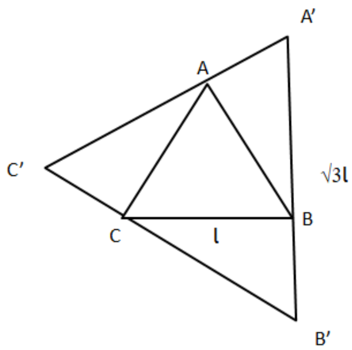
1. **Triangle**

   For a regular triangle θ = 60°

   $l\left(cot\frac{60°}{2}\right)^n = l(\sqrt{3})^n$; twist angle=n|60-90|=n*30

   Here, r>1. So, Daughter polygons get bigger and bigger.

ILLUSTRATION



2. **Square**

   For a square θ = 90

   $l\left(cot\frac{90°}{2}\right)^n$ = ; twist angle = n*|90-90|=0

   Here, r=1. So, daughter squares are of the same length and coincide with Original Square.

ILLUSTRATION

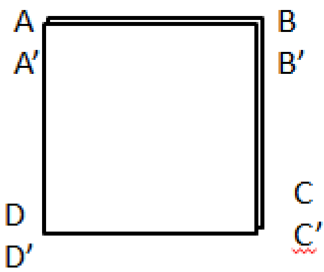


Table Showing Common ratio of different Polygons.

S.N	Regular Polygon	No. of Sides (r)	Interior angle(θ) $\left(\frac{r-2}{r}\right) \times 180°$	common ratio. $cot\frac{\theta}{2}$	
1.	Triangle	3	60°	$\sqrt{3}$	D
2	Square	4	90°	1	E
3	Pentagon	5	108°	0.762	C
4	Hexagon	6	120°	$\frac{1}{\sqrt{3}}$	R
5	Heptagon	7	128.57°	0.481	E
6	. . .	.	.	.	A
7 8 9 .	and…so….on.	. . . .	. . . .	. . . .	S I N G

From the table above, it can be concluded that the common ratio of higher polygons is smaller. For instance, If a Pentagon and Hexagon of the same side length are taken, then the side length of the daughter pentagon is greater than that of the daughter hexagon.

III. CONCLUSION:

Hence, the above-mentioned theorem can be used to determine and compare the side length of the daughter polygon for any given regular polygon.