International Journal of Multidisciplinary and Current Educational Research (IJMCER)
ISSN: 2581-7027 ||Volume|| 4 ||Issue|| 5 ||Pages 66-77 ||2022||

Hybrid Scheme with Constructed Orthogonal Polynomial for Solving Initial Value Problem of Third Order Ordinary Differential Equations.

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ABSTRACT: We considered the development of solution for initial value problems of third order ordinary differential equations, with constructed orthogonal polynomial of weight function $w(x) = x$ in the interval $[0,1]$. It was used as our basis function in a collocation and interpolation technique. By investigating the basic properties, from the findings it shows that the method is accurate and convergent. We considered three examples, the results obtained when compared with existing method are favourable.

KEYWORD: Orthogonal polynomial, Hybrid, Interpolation, Collocation, Block Method

I. INTRODUCTION

Many problems in Science and Engineering yields Initial Value Problems (IVPs) of third order ordinary differential equations (ODEs) as shown below:

$$y''' = f(x,y,y',y''); y(a) = \alpha, y'(a) = \beta, y''(a) = \gamma \quad (1)$$

where f is continuous in $[a,b]$ arises in many areas of physical problems. Some of these problems have no analytical solution, thereby numerical schemes are developed to solve the problems. The method of reducing (1) to a system of first order differential equations has been reported to increase the dimension of the problem and therefore results in more computations (see Bun (1992)). Milne (1953), proposed Block method for ODEs. Many researchers used different orthogonal polynomials as the basis function to solve the problems numerically. Chebyshev orthogonal polynomial was used by Lancsos (1983) also Tanner (1979) and Dahlguist (1979). Adeniyi, Alabi and Folaranmi (2008), Adeyefa, Akinola, Folaranmi and Owolabi (2016), Joseph, Adeniyi and Adeyefa (2018), all of these researchers constructed orthogonal polynomials in certain interval for different weight functions. In this work, an orthogonal polynomial constructed for the interval $[0,1]$ with respect to the weight function $w(x) = x$ is adopted to solve third order ODEs for the Initial Value Problem (1).

II. THE ORTHOGONAL POLYNOMIAL CONSTRUCTION

For the equation below

$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx = h_n\delta_{mn} \quad (2)$$

with

$$\delta_{mn} = \begin{cases} 0, m \neq n \\ 1, m = 1 \end{cases}$$

$w(x)$ is continuous and positive in the interval $[a, b]$ such that the moments

$$\mu = \int_a^b w(x)x^n dx,\ n = 0,1,2, \dots \quad (3)$$

exists.

The integral

$$\langle \phi_m, \phi_n \rangle = \int_a^b w(x)\phi_m(x)\phi_n(x)dx \quad (4)$$

is the inner product of the polynomials ϕ_m and ϕ_n. For orthogonality,

$$\langle \phi_m, \phi_n \rangle = \int_a^b w(x)\phi_m(x)\phi_n(x)dx = 0,\ m \neq n, [-1,1]. \quad (5)$$

For orthogonal polynomials valid in [0,1] with respect to $w(x) = x$ as the basis function $\phi_n(x)$, $n = 1,2,3, \ldots$ of the approximant

$$y_n(x) = \sum_{j=0}^{n} a_j \phi_j(x) \approx y(x). \tag{6}$$

For this purpose, let $\phi_n(x)$ be a polynomial of the nth order defined by

$$\phi_n(x) = \sum_{j=0}^{n} c_j^{(n)} x^j \tag{7}$$

The requirements for the construction are that

$$\phi_n(1) = 1 \tag{8}$$

and

$$\int_o^1 w(x)\phi_m \phi_n(x)dx = 0;\ m \neq n.$$

Using the conditions above, the following orthogonal polynomial were generated

$$\phi_o(x) = 1$$
$$\phi_1(x) = 3x - 2$$
$$\phi_2(x) = 10x^2 - 12x + 3$$
$$\phi_3(x) = 35x^3 - 60x^2 + 30x - 4$$
$$\phi_4(x) = 126x^4 - 280x^3 + 210x^2 - 60x + 5$$
$$\phi_5(x) = 462x^5 - 1260x^4 + 1260x^3 - 560x^2 + 105x - 6$$
$$\phi_6(x) = 1716x^6 - 5544x^5 + 6930x^4 - 4200x^3 + 1260x^2 - 168x + 7$$
$$\phi_7(x) = 6435x^7 - 24024x^6 + 36036x^5 + 27720x^4 + 11550x^3 - 2520x^2 + 252x - 8$$

III. DEVELOPMENT OF TWO-STEP METHOD WITH $x_{n+\frac{1}{3}}$ AS THE OFF-STEP POINT

To achieve this, the analytic solution of (1) is approximated by the trial solution of the form

$$\bar{y}(x) = \sum_{j=0}^{r+s-1} a_j \phi_j(x) \approx y(x) \tag{9}$$

where $x \in [a, b]$, r and s are the number of collocation and interpolation points respectively. The function $\phi_j(x)$ is the j^{th} degree orthogonal polynomial valid in the range of integration of $[a, b]$. The third derivative of (9) is given by

$$\bar{y}'''(x) = \sum_{j=0}^{r+s-1} a_j \phi_j'''(x) = f(x, \bar{y}, \bar{y}', \bar{y}'') \tag{10}$$

The system of equations gotten from above will be solved by Gaussian Elimination Method. In deriving this method, set $s = 3$ and $r = 4$ in (9) and (10) to give two equations each of degree six as follows.

$$\sum_{j=0}^{6} a_j \phi_j(x) = \bar{y}(x) \approx y(x) \tag{11}$$

$$\sum_{j=0}^{6} a_j \phi_j'''(x) = f(x, \bar{y}, \bar{y}', \bar{y}'') \tag{12}$$

Interpolate (11) at $x = x_{n+s}, s = 0, \frac{1}{3}, 1$; and collocate (12) at $x = x_{n+s}, s = 0, \frac{1}{3}, 1, 2$; to get the system of equations:

$$\begin{bmatrix} 1 & -5 & 25 & -129 & 681 & -3693 & 19825 \\ 1 & -4 & \frac{139}{9} & -\frac{1648}{27} & \frac{6647}{27} & -\frac{20159}{20} & \frac{70942}{17} \\ 1 & -2 & 3 & -4 & 5 & -6 & 7 \\ 0 & 0 & 0 & 210 & -4704 & 65520 & -730080 \\ 0 & 0 & 0 & 210 & -3696 & 40040 & -344933 \\ 0 & 0 & 0 & 210 & -1680 & 7560 & 25200 \\ 0 & 0 & 0 & 210 & 1344 & 5040 & 14400 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\frac{1}{3}} \\ y_{n+1} \\ h^3 f_n \\ h^3 f_{n+\frac{1}{3}} \\ h^3 f_{n+1} \\ h^3 f_{n+2} \end{bmatrix} \tag{13}$$

System (1) is solved to obtain the values of the unknown parameters $a_j, j = 0(1)6$ as follows:

$$\left.\begin{aligned}
a_0 &= \frac{17}{6} y_n - \frac{21}{4} y_{n+\frac{1}{3}} + \frac{41}{12} y_n - \frac{205}{14441} h^3 f_n + \frac{697}{5966} f_{n+\frac{1}{3}} + \frac{822}{4717} h^3 f_{n+1} + \frac{97}{35488} f_{n+2} \\
a_1 &= \frac{28}{15} y_n - \frac{33}{10} y_{n+\frac{1}{3}} + \frac{43}{30} y_n - \frac{75}{17488} h^3 f_n + \frac{157}{2385} f_{n+\frac{1}{3}} + \frac{80}{539} h^3 f_{n+1} + \frac{61}{12571} f_{n+2} \\
a_2 &= \frac{3}{10} y_n - \frac{91}{20} y_{n+\frac{1}{3}} + \frac{3}{20} y_{n+1} + \frac{23}{5407} h^3 f_n - \frac{21}{35398} f_{n+\frac{1}{3}} + \frac{101}{2074} h^3 f_{n+1} + \frac{32}{8019} f_{n+2} \\
a_3 &= h^3 \left(\frac{4}{2079} f_n - \frac{17}{3850} f_{n+\frac{1}{3}} + \frac{13}{2310} f_{n+1} + \frac{45}{27679} f_{n+2} \right) \\
a_4 &= h^3 \left(\frac{5}{133056} f_n - \frac{1}{24640} f_{n+\frac{1}{3}} - \frac{5}{14784} f_{n+1} + \frac{4}{11723} f_{n+2} \right) \\
a_5 &= h^3 \left(\frac{5}{72072} f_n - \frac{9}{57200} f_{n+\frac{1}{3}} + \frac{45}{364409} f_{n+1} - \frac{8}{225225} f_{n+2} \right) \\
a_6 &= h^3 \left(\frac{1}{137280} f_n - \frac{3}{228800} f_{n+\frac{1}{3}} + \frac{1}{137280} f_{n+1} - \frac{1}{686400} f_{n+2} \right)
\end{aligned}\right\} \tag{14}$$

Substituting (14) in (11) yields a continuous implicit two-step method in the form

$$\bar{y}(x) = \sum_{j=0}^{1} \alpha_j(x) y_{n+j} + \alpha_{\frac{1}{3}}(x) y_{n+\frac{1}{3}} + h^3 \left(\sum_{j=0}^{2} \beta_j(x) f_{n+j} + \beta_{\frac{1}{3}}(x) f_{n+\frac{1}{3}} \right) \tag{15}$$

Remark: It is to be noted here and elsewhere that $y_k = \bar{y}(x_k)$ for various values of k

From (15), by letting $t = \frac{x - x_k - h}{h}$, the parameters α 's and β 's are given by

$$\left.\begin{aligned}
\alpha_0(t) &= 3t^2 + 2t \\
\alpha_{\frac{1}{3}}(t) &= -\left(\frac{9}{2} t^2 + \frac{9}{2} t \right) \\
\alpha_1(t) &= \frac{3}{2} t^2 + \frac{5}{2} t + 1 \\
\beta_0(t) &= -h^3 \left(\frac{t^6}{80} - \frac{t^5}{120} - \frac{t^4}{24} + \frac{t^3}{1060994104080050020} \right. \\
&\quad \left. + \frac{229}{6480} t^2 + \frac{47}{3240} t + \frac{1}{509507941252785020} \right) \\
\beta_{\frac{1}{3}}(t) &= h^3 \left(\frac{9t^6}{400} + \frac{t^5}{53059255636142904} - \frac{9t^4}{80} + \frac{1}{70038217430708624} + \frac{161t^2}{900} + \frac{4t}{45} \right. \\
&\quad \left. + \frac{14637028087683264}{140} \right) \\
\beta_1(t) &= -h^3 \left(\frac{t^6}{80} + \frac{t^5}{60} - \frac{t^4}{16} - \frac{t^3}{6} - \frac{56t^2}{405} - \frac{31t}{810} + \frac{t^3}{20415647391654448} \right) \\
\beta_2(t) &= h^3 \left(\frac{t^6}{400} + \frac{t^5}{120} + \frac{t^4}{120} - \frac{1}{94750531357737664} - \right. \\
&\quad \left. \frac{64t^2}{15829} - \frac{t}{648} - \frac{1}{634897349135472640} \right)
\end{aligned}\right\} \tag{16}$$

By evaluating (15) at x_{n+2}, the main method is obtained as:

$$y_{n+2} = 5y_n - 9y_{n+1} + \frac{h^3}{12960}\left(-160f_n + 23014f_{n+\frac{1}{3}} + 49410f_{n+1} + 176f_{n+2}\right) \tag{16}$$

Differentiating (15) gives the continuous coefficients:

$$\left.\begin{aligned}
\alpha_0'(t) &= \frac{6t+2}{h} \\
\alpha_{\frac{1}{3}}'(t) &= -\frac{\left(9t+\frac{9}{2}\right)}{h} \\
\alpha_1'(t) &= \frac{3t+\frac{5}{2}}{h} \\
\beta_0'(t) &= -h^2\left(\frac{3t^5}{40} - \frac{t^4}{24} - \frac{t^3}{6} + \frac{t^2}{35366471369916676} + \frac{229t}{3240} + \frac{47}{3240}\right) \\
\beta_{\frac{1}{3}}'(t) &= h^2\left(\frac{27t^5}{200} + \frac{t^4}{10611851127228580} - \frac{9t^3}{20} + \frac{t^2}{2334607247992876} + \frac{161t}{450} + \frac{4}{45}\right) \\
\beta_1'(t) &= -h^2\left(\frac{3t^5}{40} + \frac{t^4}{12} - \frac{t^3}{4} - \frac{t^2}{2} - \frac{112t}{405} - \frac{31}{810}\right) \\
\beta_2'(t) &= h^2\left(\frac{3t^5}{200} + \frac{t^4}{24} + \frac{t^3}{30} - \frac{t^2}{31583510452579220} - \frac{131t}{16200} - \frac{1}{648}\right)
\end{aligned}\right\} \tag{17}$$

The second derivatives of continuous function (15) yield continuous coefficients:

$$\left.\begin{aligned}
\alpha_0''(t) &= \frac{6}{h^2} \\
\alpha_{\frac{1}{3}}''(t) &= \frac{-9}{h^2} \\
\alpha_1''(t) &= \frac{3}{h^2} \\
\beta_0''(t) &= -h\left(\frac{3t^4}{8} - \frac{t^3}{6} - \frac{t^2}{2} + \frac{t}{17683235680008338} - \frac{1}{850749}\right) \\
\beta_{\frac{1}{3}}''(t) &= h\left(\frac{27t^4}{40} + \frac{t^3}{2652962781807145} - \frac{27t^2}{20} + \frac{t}{11673036239951438} + \frac{161}{450}\right) \\
\beta_1''(t) &= -h\left(\frac{3t^4}{8} + \frac{t^3}{3} - \frac{3t^2}{4} - t - \frac{112}{405}\right) \\
\beta_2''(t) &= h1\left(\frac{3t^4}{40} + \frac{t^3}{6} + \frac{t^2}{10} - \frac{t}{15791755226289610} - \frac{131}{16200}\right)
\end{aligned}\right\} \tag{18}$$

The additional methods to be coupled with the main method (17) are obtained by evaluating the first and second derivatives of (15) at $x_n, x_{n+\frac{1}{3}}, x_{n+1}, x_{n+2}$ to get:

$$hy_n' + 4y_n - \frac{9}{2}y_{n+\frac{1}{3}} + \frac{1}{2}y_{n+1} = h^3\left(\frac{1}{162}f_n + \frac{83}{1800}f_{n+\frac{1}{3}} + \frac{11}{3240}f_{n+1} - \frac{1}{8100}f_{n+2}\right) \tag{20}$$

$$hy_{n+\frac{1}{3}}' + 2y_n - \frac{3}{2}y_{n+\frac{1}{3}} - \frac{1}{2}y_{n+1} = h^3\left(\frac{13}{9720}f_n - \frac{23}{675}f_{n+\frac{1}{3}} - \frac{11}{2430}f_{n+1} + \frac{11}{48600}f_{n+2}\right) \tag{21}$$

$$hy_{n+1}' - 2y_n + \frac{9}{2}y_{n+\frac{1}{3}} - \frac{5}{2}y_{n+1} = h^3\left(-\frac{47}{3240}f_n + \frac{4}{45}f_{n+\frac{1}{3}} + \frac{31}{810}f_{n+1} - \frac{1}{648}f_{n+2}\right) \tag{22}$$

$$hy_{n+2}' - 8y_n + \frac{27}{2}y_{n+\frac{1}{3}} - \frac{11}{2}y_{n+1} = h^3\left(\frac{13}{270}f_n + \frac{79}{600}f_{n+\frac{1}{3}} + \frac{979}{1080}f_{n+1} + \frac{217}{2700}f_{n+2}\right) \tag{23}$$

$$h^2y_n'' - 6y_n + 9y_{n+\frac{1}{3}} - 3y_{n+1} = h^3\left(-\frac{91}{810}f_n - \frac{571}{1800}f_{n+\frac{1}{3}} - \frac{49}{3240}f_{n+1} + \frac{1}{4050}f_{n+2}\right) \quad (24)$$

$$h^2y_{n+\frac{1}{3}}'' - 6y_n + 9y_{n+\frac{1}{3}} - 3y_{n+1} = h^3\left(\frac{91}{3240}f_n - \frac{49}{450}f_{n+\frac{1}{3}} - \frac{13}{405}f_{n+1} + \frac{29}{16200}f_{n+2}\right) \quad (25)$$

$$h^2y_{n+1}'' - 6y_n + 9y_{n+\frac{1}{3}} - 3y_{n+1} = h^3\left(-\frac{229}{3240}f_n + \frac{161}{450}f_{n+\frac{1}{3}} + \frac{112}{405}f_{n+1} - \frac{131}{16200}f_{n+2}\right) \quad (26)$$

$$h^2y_{n+2} - 6y_n + 9y_{n+\frac{1}{3}} - 3y_{n+1} = h^3\left(\frac{179}{810}f_n - \frac{571}{1800}f_{n+\frac{1}{3}} + \frac{2121}{1609}f_{n+1} + \frac{450}{1349}f_{n+2}\right) \quad (27)$$

Equations (17) and (20) − (27) are solved using Shampine and Watts (1969) block formula defined as

$$Ay_m = hBF(y_m) + E_{y_n} + hDf_n \quad (28)$$

$A = (a_{ij}), B = (b_{ij})$, column vectors $E = (e_1 \dots e_r)^T, D = (d_1 \dots d_r)^T$, $y_m = (y_{n+1} \dots y_{n+r})^T$ and $f(y_m) = (f_{n+1}, \dots, f_{n+r})^T$

This leads to the matrices:

$$A = \begin{bmatrix} 9 & -5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{9}{5} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{9}{2} & -\frac{5}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{27}{2} & -\frac{11}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 9 & -3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 9 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \frac{8}{45} & \frac{61}{16} & \frac{11}{810} \\ \frac{83}{1000} & \frac{11}{3240} & \frac{-1}{8100} \\ \frac{23}{615} & \frac{-11}{2430} & \frac{11}{48600} \\ \frac{4}{45} & \frac{31}{810} & \frac{-1}{648} \\ \frac{79}{600} & \frac{979}{1080} & \frac{217}{2700} \\ \frac{-571}{1800} & \frac{-9}{3240} & \frac{1}{4050} \\ \frac{-49}{450} & \frac{-13}{405} & \frac{24}{16200} \\ \frac{161}{450} & \frac{112}{405} & \frac{-131}{1620} \\ \frac{-51}{1800} & \frac{2121}{1609} & \frac{450}{1349} \end{bmatrix}, E = \begin{bmatrix} 5 & 0 & 0 \\ 4 & 0 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 0 \\ 8 & 0 & 0 \\ 6 & 0 & 0 \\ 6 & 0 & 0 \\ 6 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix}.$$

$$D = \left(-\frac{1}{82}\ \frac{1}{162}\ \frac{13}{9720}\ \frac{-47}{3240}\ \frac{13}{270}\ \frac{-19}{810}\ \frac{91}{3240}\ \frac{-229}{3240}\ \frac{179}{810}\right)^T$$

Substituting A, B, D and E into equation (28) yields the following explicit schemes:

$$y_{n+\frac{1}{3}} = y_n + \frac{h}{3}y_n' + \frac{h^2}{18}y_n'' + \frac{61h^3}{14580}f_n + \frac{73h^3}{32400}f_{n+\frac{1}{3}} - \frac{17h^3}{58320}f_{n+1} + \frac{h^3}{36450}f_{n+2} \quad (29)$$

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{20}f_n + \frac{9h^3}{80}f_{n+\frac{1}{3}} + \frac{h^3}{240}f_{n+1} \quad (30)$$

$$y_{n+2} = y_n + 2y_n'h + 2h^2y_n'' + \frac{h^3}{5}f_n + \frac{18h^3}{25}f_{n+\frac{1}{3}} + \frac{1707h^3}{445}f_{n+1} + \frac{h^3}{5}f_{n+2} \quad (19)$$

$$y_{n+\frac{1}{3}}' = y_n' + \frac{h}{3}y_n'' + \frac{317h^2}{9720}f_n + \frac{23h^2}{900}f_{n+\frac{1}{3}} - \frac{7h^2}{2430}f_{n+1} + \frac{13h^2}{48600}f_{n+2} \quad (32)$$

$$y_{n+1}' = y_n' + y_n'' + \frac{11h^2}{120}f_n + \frac{9h^2}{25}f_{n+\frac{1}{3}} + \frac{h^2}{20}f_{n+1} - \frac{h^2}{600}f_{n+2} \quad (33)$$

$$y'_{n+2} = y'_n + 2hy''_n + \frac{4h^2}{15}f_n + \frac{18h^2}{25}f_{n+\frac{1}{3}} + \frac{14h^2}{15}f_{n+1} + \frac{2h^2}{25}f_{n+2} \tag{34}$$

$$y''_{n+\frac{1}{3}} = y''_n + \frac{91h}{648}f_n + \frac{5h}{24}f_{n+\frac{1}{3}} - \frac{11h}{648}f_{n+1} + \frac{h}{648}f_{n+2} \tag{35}$$

$$y''_{n+1} = y''_n + \frac{h}{24}f_n + \frac{27h}{40}f_{n+\frac{1}{3}} + \frac{7h}{24}f_{n+1} - \frac{h}{120}f_{n+2} \tag{36}$$

$$y''_{n+2} = y''_n + \frac{h}{3}f_n + \frac{4h}{3}f_{n+1} + \frac{h}{3}f_{n+2} \tag{37}$$

IV. ANALYSIS OF THE NEW METHODS

The main methods derived are discrete schemes belonging to the class of LMMs of the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j f_{n+j} \tag{38}$$

Following Futunla (1988) and Lambert (1973), we define the Local Truncation Error (LTE) associated with (38) by difference operator;

$$L[y(x):h] = \sum_{j=0}^{k} \left[\alpha_j y(x_n + jh) - h^3 \beta_j f(x_n + jh)\right] \tag{39}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a, b]$.
Expanding (38) in Taylor's Series about the point x, we obtain the expression

$$L[y(x):h] = c_o y(x) + c_1 h y'(x) + \cdots c_{p+3} h^{p+3} y^{p+3}(x) \tag{40}$$

where the $c_o, c_1, c_2 \ldots c_p \ldots c_{p+3}$ are obtained

$$c_0 = \sum_{j=0}^{k} \alpha_j \tag{41}$$

$$c_1 = \sum_{j=1}^{k} j\alpha_j \tag{42}$$

$$c_3 = \frac{1}{3!}\sum_{j=1}^{k} j^3 \alpha_j \tag{43}$$

$$c_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q \alpha_j - q(q-1)(q-2)(q-3)\sum_{j=1}^{k} \beta_j j^{q-3}\right] \tag{44}$$

In the sense of Lambert (1973), equation (38) is of order p if $c_o = c_1 = c_2 = c_2 = \cdots c_p = c_{p+1} = c_{p+2} = 0$ and $c_{p+3} \neq 0$. The $c_{p+3} \neq 0$ is called the error constant and $c_{p+3} h^{p+3} y^{p+3}(x_n)$ is the Principal Local truncation error at the point x_n. The equation (17) have order p $= 4$ and error constants $C_{p+3} = \frac{-5}{11664}$.

Zero stability : The LMM (38) is said to be Zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if and only if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyze the Zero-stability of the method, we present equations (29) -(37).in the block form

$$A^0 y_m = hBF(y_m) + A' y_n hD f_n$$

where h is a fixed mesh size within a block. In line with these, equations (29) - (37) give

$$A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 73/32400 & -17/58320 & 1/36450 \\ 9/80 & 1/240 & 0 \\ 18/25 & 1707/445 & 1/75 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 61/14580 \\ 0 & 0 & 1/20 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

The first characteristic polynomial of the block hybrid method is given by

$$\rho(R) = \det(RA^0 - A') \tag{45}$$

Substituting A^0 and A' in equation (44) and solving for R, the values of R are obtained as 0 and 1. According to Fatunla (1988,1991), the block method equations (29)-(37) are zero-stable, since from (45), $\rho(R) = 0$, satisfy $|R_j| \leq 1, j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed three.

1.0.2 Consistency

The LMM (1) is said to be consistent if it has order $p \geq 1$ and the first and second characteristic polynomials which are defined respectively, as

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j \tag{46}$$

$$\sum_{j=0}^{k} \alpha_j = 0$$

(47)

where z is the principal root, satisfy the following conditions:

$$\sum_{j=0}^{k} \alpha_j = 0 \tag{48}$$

$$\rho(1) = \rho'(1) = 0 \tag{49}$$

and

$$\rho'''(1) = 3!\,\sigma(1) \tag{50}$$

(Henrichi, 1962)

The scheme (17) is of order $\rho = 4 > 1$ and have been investigated to satisfy conditions (I)-(III) Hence, the scheme is consistent.

Convergence : By the theorem of Dahlguist, the necessary and sufficient condition for an LMM to be convergent, is that, it is consistent and zero-stable. The methods satisfy the two conditions stated in definition above and hence the method is convergent.

Region of Absolute Stability (RAS)

For the two-step method with Off-step Point $\frac{1}{3}$, we have

$$y_{n+2} - 5y_n - 9y_{n+1} = \frac{h^3}{12960}\left(-160f_n + 23014f_{n+\frac{1}{3}} + 49410f_{n+1} + 176f_{n+2}\right)$$

$$\bar{h}(z) = \frac{12960(z^2 + 9z - 5)}{23014z^{\frac{1}{3}} + 49410z + 176z^2 - 160}$$

$$\bar{h}(\theta) = \frac{12960e^{i2\theta} + 9e^{i\theta} - 5}{23014e^{i\frac{1}{3}\theta} + 49410e^{i\theta} + 176e^{i2\theta} - 160}$$

The RAS is shown in the figure below

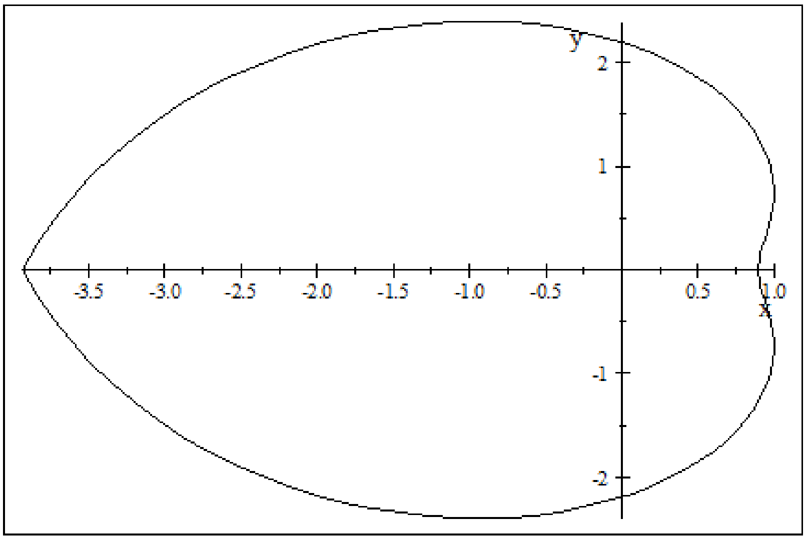

Figure 1: Region of Absolute Stability for Two-step with Off-step Point $\frac{1}{3}$

V. APPLICATION OF THE METHODS

Three problems will be considered in this section.

Problem 1

Consider the constant-coefficient non-homogeneous problem

$$y''' + y'' + 3y' - 5y = 2 + 6x - 5x^2, 0 \leq x \leq 1$$

$$y(0) = -1, y'(0) = 1, y''(0) = -3$$

sourced from Awoyemi et al (2014) and whose exact solution is $y(x) = x^2 - e^x + e^x \sin 2x$.

This was solved using step length $h = 0.1$.

Problem 2

Here, the constant coefficient homogeneous problem sourced from Anake et al (2013):

$$y''' + y' = 0$$

$$y(0) = 0, y'(0) = 1, y''(0) = 2$$

whose analytic solution is $y(x) = 2(1 - \cos x) + \sin x$

was solved with step size $h = 0.1$.

Problem 3

The stiff problem

$$y''' + 100y'' + y = 102e^x + e^{-x}$$

$$y(0) = 2, y'(0) = -99, y''(0) = 1001$$

whose true solution is $y(x) = e^x + e^{-100x}$ will be solved over step size $h = 0.00001$.

VI. TABLES OF RESULTS

Table 1: Results for Problem 1

x	Exact solution	Two-steps with $v = \frac{1}{3}$
0.1	-0.915407473756115	-0.915407472777346
0.2	-0.862573985499430	-0.862573921968956
0.3	-0.841561375114166	-0.841561261760400
0.4	-0.850966529765556	-0.850966621357168
0.5	-0.888343319155557	-0.888344202788101
0.6	-0.950604904717256	-0.950607522086856

0.7	−1.034392853933000	−1.034398467952800
0.8	−1.136403556878910	−1.136413676460480
0.9	−1.253666211231610	−1.253682488273390
1.0	−1.383769999219790	−1.383794111055600

Table 2: Results for Problem 2

x	Exact solution	Two-steps with $v = \frac{1}{3}$
0.1	0.109825086090778	0.109825086087205
0.2	0.238536175112581	0.238536175212721
0.3	0.384847228410130	0.384847228898569
0.4	0.547296354302880	0.547296355663045
0.5	0.724260414823453	0.724260417755140
0.6	0.913971243575675	0.913971249008162
0.7	1.114533312668710	1.114533321769150
0.8	1.323942672205190	1.323942686381960
0.9	1.540106973086150	1.540106993987070
1.0	1.760866373071620	1.760866402576810

Table 3: Results for Problem 3

x	Exact solution	Two-steps with $v = \frac{1}{3}$
0.00001	1.990149838749340	1.990104988375080
0.00002	1.980398693308080	1.980219887332010
0.00003	1.970745578553010	1.970344598359350
0.00004	1.961189519162990	1.960479023925900
0.00005	1.951729549521550	1.950623067470910
0.00006	1.942364713620260	1.940776633394440
0.00007	1.933094064963130	1.930939627047790

4 Tables of Errors

Table 4: Error of the Methods for Problem 1

x	Two-steps with $v = \frac{1}{3}$	ERROR IN AWOYEMI
0.1	9.78769×10^{-10}	6.408641×10^{-7}
0.2	6.3530474×10^{-8}	1.51133×10^{-5}

0.3	$1.13353766 \times 10^{-8}$	6.364443×10^{-5}
0.4	9.1591612×10^{-8}	1.675667×10^{-4}
0.5	$8.83632544 \times 10^{-8}$	3.507709×10^{-4}
0.6	2.6173696×10^{-5}	6.410875×10^{-4}
0.7	5.6140198×10^{-5}	1.071642×10^{-3}
0.8	$1.011958157 \times 10^{-4}$	1.682213×10^{-3}
0.9	$1.627704178 \times 10^{-4}$	2.520603×10^{-3}
1.0	$2.411183581 \times 10^{-4}$	3.644014×10^{-3}

Table 5: Error of the Methods for Problem 2

x	Two-steps with $v = \frac{1}{3}$	ERROR IN Anake (2013)
0.1	3.573×10^{-12}	1.6088×10^{-9}
0.2	1.0014×10^{-10}	1.0387×10^{-8}
0.3	4.88439×10^{-10}	2.9572×10^{-8}
0.4	1.360165×10^{-9}	2.3147×10^{-7}
0.5	2.931687×10^{-9}	4.542×10^{-7}
0.6	5.432487×10^{-9}	1.4746×10^{-6}
0.7	$9.10044X10^{-9}$	$2.8734X10^{-6}$
0.8	1.417677×10^{-8}	4.6826×10^{-6}
0.9	2.090092×10^{-8}	6.9217×10^{-6}
1.0	2.950519×10^{-8}	9.5974×10^{-6}

Table 6: Error of the Methods for Problem 3

x	Two-steps with $v = \frac{1}{3}$
0.00001	$4.485037426 \times 10^{-4}$
0.00002	$1.7880597607 \times 10^{-3}$
0.00003	$4.0098019366 \times 10^{-3}$
0.00004	$7.1049523709 \times 10^{-3}$
0.00005	$1.10648205064 \times 10^{-2}$
0.00006	$1.58808022582 \times 10^{-2}$
0.00007	$2.15443791534 \times 10^{-2}$

VII. CONCLUSION

Continuous hybrid scheme with off point $v = \frac{1}{3}$ was used with constructed orthogonal polynomials as basis function in collocation and interpolation technique. The method was analyzed, and shown to be consistent and zero stable, and hence convergent. Three selected problems have been considered to test the effectiveness and accuracy of the method. From our table of results, it is clear that the method is accurate and effective since the approximation closely estimate the analytic solution.

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