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Linear Multistep Method of Second Derivative Block Hybrid Method for the Solution of Stiff System of Ordinary Differential Equations

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ABSTRACT: In this paper work, the Construction of Linear Multistep Method of Second Derivative Block Hybrid for the Solution of Stiff System of Ordinary Differential Equations of general first order initial value problems is studied. In the derivation of the multistep collocation approach using the matrix inversion method of Onumanyi (1994) and Sirisena (1997), Power series is adopted as basis function to obtain the main discrete and continuous scheme through collocation and interpolations approach. The second derivatives linear multi-step block Hybrid method which incorporated with two off grid points for step number k = 2 through multi-step collocation of the new method is consistent, convergent and zero stable. The efficiency of the new method was tested on some stiff system of equations and found to give better approximation than the existing method.

KEYWORDS: Linear Multi-step method, Hybrid block, Stiff ODEs, Interpolation, implicit, power series, matrix inversion and Collocation.

I. INTRODUCTION

We consider the numerical solution of first and second order differential equation for the initial value problems of the form:

$$y' = f(x, y),\ y(a) = y_0 \tag{1.1}$$

$$y'' = f(x, y, y') \tag{1.2}$$

Where f is continuous and satisfies Lipchitz's condition that guarantees the uniqueness and existence of a solution. General linear methods emerged because of the desire to obtain a wider generalization of a large family of traditional numerical methods for ordinary differential equations were first introduced by Butcher (1966), as a unifying theory for studying stability, consistency and convergence for a wide variety of traditional methods. The transformation of k-step multi-step method continue form and evaluation at various mesh point to obtain discrete schemes. The drives discrete schemes applied simultaneously as block for moving integration process forward with k-step at a time. Donald, Yusuf, Pius, and Paul (2009).In this paper, we developed a two-step second derivative hybrid block method for direct solution of seconds Order Ordinary differential equations, which is implemented in block method. The method developed evaluates less function per step and circumventing the Dahlquist barrier's by introducing a hybrid points. The paper is organised as follows: In section 2, we discuss the methods and the materials for the development of the method. Section 3 considers analysis of the basis properties of the method, which include convergence and stability region, numerical experiments where the efficiency of the derived method is tested on some stiff numerical examples and discussion of results. Raymond, Skwame and Kyagya (2018). And the fourth section, some numerical problems were solved and the performance of the developed method was compared with those of the existing methods, Donald, Skwame, Dedan (2018), finally, our conclusion was drawn in section five.

II. DERIVATION OF THE METHOD

We consider a power series approximate solution of the form

$$y(x) = \sum_{i=0}^{2v+m-1} a_i \left(\frac{x - x_n}{h}\right)^i, \tag{2.1}$$

Gives the following nonlinear equations which can be written in matrix form as the approximate solution to (1.1) for $x \in [x_n, x_{n+1}]$ where $n = 0, 1, 2, \cdots N-1$, $a's$ are the real coefficients to be determined, v is the number of collocation points, m is the number of interpolation points and $h = x_n - x_{n-1}$ is a constant step size of the partition of interval $[a, b]$which is given by $a = x_0 < x_1 < \cdots < x_N = b$.

By modification of the matrix inversion approach (1.1) of Sirisena (1997), for second derivative, a k-step multistep collocation method with m collocation points by differentiating (1.2) once and twice yields becomes:

$$y'(x) = \sum_{i=1}^{2v+m-1} \frac{ia_i}{h}\left(\frac{x-x_n}{h}\right)^{i-1} = f(x,y) \qquad (2.2)$$

$$y''(x) = \sum_{i=2}^{2v+m-1} \frac{i(i-1)a_i}{h^2}\left(\frac{x-x_n}{h}\right)^{i-2} = g(x,y,y') \qquad (2.3)$$

Where v and m are the numbers of collocation and interpolation points respectively, $a_j{}'s$ are parameters to be determined. We consider a sequence of points $\{x_n\}$ in the interval $I=[a,b]$ defined by $a = x_0 < x_1 < \ldots < x_\eta = b$, such that $h_i = x_{i+1} - x_i$, $i = 0,1,2,\ldots N-1$.

We now consider the derivation of the multistep collocation method for constant step size h defined for the step $|x_n, x_{n+1}|$

$$y(x) = \sum_{j=0}^{m} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{v} \beta_j(x) f_{n+j} + h^2 \sum_{j=0}^{v} \gamma_j(x) g_{n+j} \qquad (2.4)$$

Such that it satisfies the conditions

$$\bar{y}(x_{n+j}) = y_{n+j},\ j \in (0,1,2,\ldots,m) \qquad (2.5)$$

$$\bar{y}'(\bar{x}_{n+j}) = f_{n+j},\ j \in (0,1,2,\ldots,v) \qquad (2.6)$$

Where m denotes the number of interpolation points x_{n+j}, $j = 0,1,2,\ldots m$ and v denotes the number of collocation points $\bar{x}_j$, $f(x_n \ldots, x_{n+1})$, $j = 0,1,2,\ldots v$.

The points x_j are chosen from the step x_{n+j} as well as one off grid point from (2.4) the coefficient polynomials are of the form

$$\alpha_j(x) = \sum_{j=0}^{m} \alpha_{j,j+1} x^j,\ j \in (o,1,2,\ldots m) \qquad (2.7)$$

$$h\beta_j(x) = h \sum_{j=0}^{s+r-1} \beta_{j,j+1} x^j,\ j \in (o,1,2,\ldots m+v-1) \qquad (2.8)$$

$$h^2\gamma_j(x) = h \sum_{j=0}^{s+r-1} \gamma_{j,j+1} x^j,\ j \in (o,1,2,\ldots m+v-1) \qquad (2.9)$$

To determine $\alpha_j(x), \beta_j(x)$ *and* $\gamma_j(x)$, Sirisena (1997) arrived at a matrix of the form

$$DC = I \qquad (2.10)$$

Where we identity matrix of dimension (m + v) x (m + v) while D and C are matrices defined as

$$D = \begin{vmatrix} 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{v+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & \cdots & x_n^{v+m-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n+m-1} & x_{n+m-1}^2 & x_{n+m-1}^3 & & x_{n+m-1}^{v+m-1} \\ 0 & 1 & 2\bar{x}_0 & 3\bar{x}_0^{v+m-1} & & (v+m-1)\bar{x}_0^{v+m-2} \\ 0 & 1 & 2\bar{x}_{m-1} & 3\bar{x}_{m-1}^{v+m-1} & & (v+m-1)\bar{x}_{m-1}^{v+m-2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 2 & 6\bar{x}_0 & & (v+m-2)(v+m-1)\bar{x}_0^{v+m-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 2 & 6\bar{x}_{m-2} & & (v+m-2)(v+m-1)\bar{x}_{m-2}^{v+m-3} \end{vmatrix} \qquad (2.11)$$

And

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots\cdots & \alpha_{s-1,1} & h\beta_{0,1} & . & h\beta_{r-1,1} & h^2\gamma_{0,1} & \cdots\cdots & \cdots & h^2\gamma_{r-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots\cdots & \alpha_{s-1,2} & h\beta_{0,2} & . & h\beta_{r-1,2} & h^2\gamma_{0,2} & \cdots\cdots & \cdots & h^2\gamma_{r-1,2} \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ \alpha_{0,s+r} & \alpha_{1,s+r} & \cdots\cdots & \alpha_{s-r,s+r} & h\beta_{0,s+r} & . & h\beta_{r-1,s+r} & h^2\gamma_{0,s+r} & . & . & h^2\gamma_{r-1,s+r} \end{bmatrix} \quad (2.12)$$

where $\overline{x_0} = x_n$, $\overline{x_1} = x_{n+1}$

From (2.10), we have that C = D^{-1}, where the columns of C give the continuous coefficients $\alpha_j(x), \beta_j(x)$ *and* $\gamma_j(x)$ of the continuous schemes

III. SPECIFICATION OF THE METHOD

The new method constructed with continues coefficients based on multistep collocation (MC). These coefficients are obtained as polynomials in terms of columns of an associated inverse matrix and are given implicitly for multi-step hybrid block method of second derivative. The values are then substituted into equation (2) to give the implicit continuous hybrid method of the form

$$p(x) = \sum_{j=0}^{v_j} \alpha_j(x) y_{n+j} + h\left[\sum_{j=0}^{k} \beta_j(x) y_{n+j} + \beta v_j(x) y n{+}v_i\right] + h^2\left[\sum_{j=0}^{k} \gamma_j(x) y_{n+j} + \gamma v_j(x) y n{+}v_i\right] \quad (3.1)$$

with first and second derivative given by

$$p'(x) = \frac{1}{h}\left[\sum_{j=0}^{v_j} \alpha_j(x) y_{n+j}\right] + \left[\sum_{j=0}^{k} \beta_j(x) y_{n+j} + \beta v_j(x) y n{+}v_i\right] + h\left[\sum_{j=0}^{k} \gamma_j(x) y_{n+j} + \gamma v_j(x) y n{+}v_i\right] \quad (3.2)$$

$$p''(x) = \frac{1}{h^2}\left[\sum_{j=0}^{v_j} \alpha_j(x) y_{n+j}\right] + \left[\sum_{j=0}^{k} \beta_j(x) y_{n+j} + \beta v_j(x) y n{+}v_i\right] + \left[\sum_{j=0}^{k} \gamma_j(x) y_{n+j} + \gamma v_j(x) y n{+}v_i\right] \quad (3.3)$$

Let express $\alpha_j(x)$ and $\beta_j(x)$ as continuous functions of t by letting

$$t = \frac{x - x_{n+v_i}}{h} \text{ and } \frac{dt}{dx} = \frac{1}{h}$$

To derive this method, two off step points is introduced. This off step points is carefully selected to guarantee zero stability condition. For this method, the off step points is $\left\{\frac{1}{2}, \frac{3}{2}\right\}$. Using (3.1) with $m=1$ and $v=5$, we have a polynomial of degree $2v+u-1$ as follows

$$y(x) = \sum_{j=0}^{10} a_j \left(\frac{x - x_n}{h}\right)^j \quad (3.4)$$

with first and second derivative given by

$$y'(x)=\sum_{j=0}^{10} ja_j\left(\frac{x-xn}{h}\right)^{j-1} \quad (3.5)$$

$$y''(x)=\sum_{j=0}^{10} j(j-1)a_j\left(\frac{x-x_n}{h}\right)^{j-2} \quad (3.6)$$

Substituting (3.6) and (3.5) into (1.1) gives

$$^j y''(x)=f\left(x,{}^j y,{}^j y'\right)=\sum_{i=1}^{10}\frac{i\,a_i}{h}\left(\frac{x-xn}{h}\right)^{i-1}+\sum_{i=2}^{10}\frac{i(i-1)a_i}{h^2}\left(\frac{x-xn}{h}\right)^{i-2},\ j=1,\ldots,m \quad (3.7)$$

Now interpolating (3.4) at $x_{n+m}, u=0$ and collocating (3.7) at $x_{n+\hat{v}}$, $\hat{v}=\left\{0,\frac{1}{2},1,\frac{3}{2},2\right\}$ leads to the system of equations written in the matrix form $AX=U$ as

$$\begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10}\\
0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 & 10x_n^9\\
0 & 1 & h+2x_n & 3(\frac{1}{2}h+x_n)^2 & 4(\frac{1}{2}h+x_n)^3 & 5(\frac{1}{2}h+x_n)^4 & 6(\frac{1}{2}h+x_n)^5 & 7(\frac{1}{2}h+x_n)^6 & 8(\frac{1}{2}h+x_n)^7 & 9(\frac{1}{2}h+x_n)^8 & 10(\frac{1}{2}h+x_n)^9\\
0 & 1 & 2h+2x_n & 3(h+x_n)^2 & 4(h+x_n)^3 & 5(h+x_n)^4 & 6(h+x_n)^5 & 7(h+x_n)^6 & 8(h+x_n)^7 & 9(h+x_n)^8 & 10(h+x_n)^9\\
0 & 1 & 3h+2x_n & 3(\frac{3}{2}h+x_n)^2 & 4(\frac{3}{2}h+x_n)^3 & 5(\frac{3}{2}h+x_n)^4 & 6(\frac{3}{2}h+x_n)^5 & 7(\frac{3}{2}h+x_n)^6 & 8(\frac{3}{2}h+x_n)^7 & 9(\frac{3}{2}h+x_n)^8 & 10(\frac{3}{2}h+x_n)^9\\
0 & 1 & 4h+2x_n & 3(2h+x_n)^2 & 4(2h+x_n)^3 & 4(2h+x_n)^4 & 6(2h+x_n)^5 & 7(2h+x_n)^6 & 8(2h+x_n)^7 & 9(2h+x_n)^8 & 10(2h+x_n)^9\\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 & 90x_n^8\\
0 & 0 & 2 & 3h+6x_n & 12(\frac{1}{2}h+x_n)^2 & 20(\frac{1}{2}h+x_n)^3 & 30(\frac{1}{2}h+x_n)^3 & 42(\frac{1}{2}h+x_n)^4 & 56(\frac{1}{2}h+x_n)^5 & 72(\frac{1}{2}h+x_n)^6 & 90(\frac{1}{2}h+x_n)^8\\
0 & 0 & 2 & 6h+6x_n & 12(h+x_n)^2 & 20(h+x_n)^3 & 30(h+x_n)^3 & 42(h+x_n)^4 & 56(h+x_n)^5 & 72(h+x_n)^6 & 90(h+x_n)^8\\
0 & 0 & 2 & 9h+6x_n & 12(\frac{3}{2}h+x_n)^2 & 20(\frac{3}{2}h+x_n)^3 & 30(\frac{3}{2}h+x_n)^3 & 42(\frac{3}{2}h+x_n)^4 & 56(\frac{3}{2}h+x_n)^5 & 72(\frac{3}{2}h+x_n)^6 & 90(\frac{3}{2}h+x_n)^8\\
0 & 0 & 2 & 12h+6x_n & 12(2h+x_n)^2 & 20(2h+x_n)^3 & 30(2h+x_n)^3 & 42(2h+x_n)^4 & 56(2h+x_n)^5 & 72(2h+x_n)^6 & 90(2h+x_n)^8
\end{pmatrix}$$

Employing the Gaussian elimination method on equation (3.6) gives the coefficient $a_j,\left(j=0(1)10\right)$. The values are then substituted into equation (3.2) and (3.3) to give the implicit continuous hybrid method of the form;

$$p(x)=\sum_{i=0}\alpha_i y_{n+i}+h\left[\sum_{i=\frac{1}{2},\frac{3}{2}}\beta_i f_{n+i}+\sum_{i=0}^{2}\beta_i f_{n+i}\right]+h^2\left[\sum_{i=\frac{1}{2},\frac{3}{2}}\gamma_i g_{n+i}+\sum_{i=0}^{2}\gamma_i g_{n+i}\right] \quad (3.8)$$

Differentiating equation (3.8) once gives;

$$p(x)=\frac{1}{h}\sum_{i=0}\alpha_i y_{n+i}+\left[\sum_{i=\frac{1}{2},\frac{3}{2}}\beta_i f_{n+i}+\sum_{i=0}^{2}\beta_i f_{n+i}\right]+h\left[\sum_{i=\frac{1}{2},\frac{3}{2}}\gamma_i g_{n+i}+\sum_{i=0}^{2}\gamma_i g_{n+i}\right] \quad (3.9)$$

where

$$\alpha_0=1$$

$$\beta_0=\frac{1}{136080}thf_n\left(50400t^9-553280t^8+2601900t^7-6832800t^6+10925250t^5-10818612t^4+6348825t^3-1833300t^2+136080\right)$$

$$\beta_{\frac{1}{2}}=\frac{4}{8505}t^3hf_{n+\frac{1}{2}}\left(5040t^7-51520t^6+220815t^5-512280t^4+690060t^3-532728t^2+211680t-30240\right)$$

$$\beta_1=\frac{1}{315}t^3hf_{n+1}\left(560t^6-5040t^5+18360t^4-34440t^3+34839t^2-17955t+3780\right)$$

$$\beta_{\frac{3}{2}} = \frac{2}{2835} t^3 h^2 g_{n+\frac{3}{2}} \left(1008 t^7 - 9520 t^6 + 37485 t^5 - 79740 t^4 + 99120 t^3 - 72324 t^2 + 28980 t - 5040\right)$$

$$\beta_2 = -\frac{1}{136080} t^3 h f_{n+2} \left(50400 t^7 - 454720 t^6 + 1714860 t^5 - 3509280 t^4 + 4218690 t^3 - 2993508 t^2 + 1172745 t - 200340\right)$$

$$\gamma_0 = \frac{1}{45360} t^2 h^2 g_n \left(2016 t^8 - 22400 t^7 + 107100 t^6 - 288000 t^5 + 477330 t^4 - 501480 t^3 + 329175 t^2 - 126000 t + 22680\right)$$

$$\gamma_{\frac{1}{2}} = \frac{2}{2835} t^3 h^2 g_{n+\frac{1}{2}} \left(1008 t^7 - 10640 t^6 + 47565 t^5 - 116820 t^4 + 170520 t^3 - 148428 t^2 + 71820 t - 15120\right)$$

$$\gamma_1 = \frac{1}{10} t^3 h^2 g_{n+1} \left(16 t^4 - 64 t^3 + 94 t^2 - 60 t + 15\right) (t-2)^3$$

$$\gamma_{\frac{3}{2}} = \frac{2}{2835} t^3 h^2 g_{n+\frac{3}{2}} \left(1008 t^7 - 9520 t^6 + 37485 t^5 - 79740 t^4 + 99120 t^3 - 72324 t^2 + 28980 t - 5040\right)$$

$$\gamma_2 = \frac{1}{45360} t^3 h^2 g_{n+2} \left(2016 t^7 - 17920 t^6 + 66780 t^5 - 135360 t^4 + 161490 t^3 - 113904 t^2 + 44415 t - 7560\right)$$

(3.10)

Evaluating the continuous schemes $y(x)$ in (3.9) at the grid points

$$x = x_n, x = x_{n+\frac{1}{2}}, x = x_{n+1}, x = x_{n+\frac{3}{2}}, x = x_{n+2}$$

To obtain the following four discrete schemes which can solve simultaneously for accurate treatment of system of first and second order differential equation, if desire

$$y_{n+\frac{1}{2}} := y_n + \frac{1}{8709120} h(153955 f_n + 1429936 f_{n+\frac{1}{2}} + 711936 f_{n+1} + 613456 f_{n+\frac{3}{2}} + 59681 f_{n+2}) + \frac{1}{2903040} h^2 (26051 g_n - 249656 g_{n+\frac{1}{2}} + 183708 g_{n+1} + 49720 g_{g+\frac{3}{2}} + 2237 g_{n+2})$$

$$y_{n+1} := y_n + \frac{1}{8136080} h(24463 f_n + 52928 f_{n+\frac{1}{2}} + 44928 f_{n+1} + 12608 f_{n+\frac{3}{2}} + 1153 f_{n+2}) + \frac{1}{45360} h^2 (421 g_n - 3040 g_{n+\frac{1}{2}} - 4536 g_{n+1} - 992 g_{g+\frac{3}{2}} - 43 g_{n+2})$$

$$y_{n+\frac{3}{2}} := y_n + \frac{3}{35840} h(2167 f_n + 4912 f_{n+\frac{1}{2}} + 6712 f_{n+1} + 3792 f_{n+\frac{3}{2}} + 137 f_{n+2}) + \frac{1}{35840} h^2 (113 g_n - 744 g_{n+\frac{1}{2}} - 756 g_{n+1} - 488 g_{g+\frac{3}{2}} - 15 g_{n+2})$$

$$y_{n+2} := y_n + \frac{1}{8505} h(1601 f_n + 4096 f_{n+\frac{1}{2}} + 5616 f_{n+1} + 4096 f_{n+\frac{3}{2}} + 1601 f_{n+2}) + \frac{1}{2835} h^2 (29 g_n - 128 g_{n+\frac{1}{2}} + 128 g_{g+\frac{3}{2}} - 29 g_{n+2})$$

(3.11)

IV. ANALYSIS OF THE METHOD

4.1 Order and error constant of our new method : The order and error constants will be defined following the method of Chollom *et.al* (2003) however, with some modification to accommodate general higher order ordinary differential equations and offstep points.

The order and error constants of the new method is obtained using

$$\ell[y(x)\,;\,h] = \sum_{j=0}^{k} \alpha_j y(x+jh) - h\sum_{j=0}^{k}\left(\beta_j y'(x+jh)\right) - h^2\sum_{j=0}^{k}\left(\gamma_j y''(x+jh)\right) \qquad (4.1.1)$$

The equation can be written in Taylor series expansion about the point x to obtain the expression

$$l[y(x):h] = c_0\, y(x) + c_1 h\, y'(x) + c_2 h^2 y''(x) + \ldots + c_{p+1} h^{p+1} y^{(p+1)}(x) + \ldots \qquad (4.1.2)$$

where the constant coefficients $C_q, q = 0,1,2,\ldots$ are given as follows

$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_k$

$c_1 = \alpha_0 + 2\alpha_1 + 3\alpha_2 + \ldots + k\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \ldots + \beta_k)$

$c_q = \frac{1}{q!}(\alpha_0 + 2^q\alpha_1 + \ldots + k^q\alpha_k) + \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + \ldots + k^{q-1}\beta_k)$, q=2,3,…

(4.1.3)

Hence, Equation (4.1.3) is of order p if,

$$l[y(x):h] = 0\left(h^{p+2}\right), C_0 = C_1 = C_2 = \ldots = C_{p+1} = 0,\quad C_{p+2} \neq 0$$

(4.1.4)

The truncation error is then given as $C_{p+2} = 11$ *in which* $p = 9$, comparing the coefficient of h gives

$C_0 = C_1 = C_2 = C_3 = \ldots = C_{10} = 0$ and $C_{11} = \left[\frac{551}{643778150400}, \frac{1}{10059033600}, \frac{1}{883097600}, \frac{1}{502951690}\right]^T$

4.2 Consistency of the Method : The hybrid block method is said to be consistent if it has an order more than or equal to one i.e. $P \geq 1$. Therefore, the new method is consistent Dahlquist (1956).

4.3 Zero Stability of the Method : The hybrid method is said to be zero stable if the first characteristic polynomial $\pi(r)$ having roots such that $|r_z| \leq 1$ *and if* $|r_z| = 1$, then the multiplicity of r_z must not be greater than two, Dahlquist(1956) and Butcher(2009) expressed in the form

$$p(z) = \left| z\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = z^3(z-1)$$

$p(z) = z^3(z-1) = 0, z = 0,0,0,1$

Hence, our method is zero-stable.

4.4 Convergence of the Method : Definition (4.1) The convergence of the continuous implicit hybrid the method is considered in the light of the basic properties discussed earlier in conjunction with the fundamental theorem of Dahiquist, Henrici (1962) for linear multistep method.

4.5 Region of Absolute Stability of the Method : By applying boundary locus condition from equation (28)

$RAS: Aw - E_0 - E_1 - dh - bhw - h^2c - h^2pw$:

$$A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ E_0=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix},E_1=\begin{bmatrix}0&0&0&\frac{1}{2}\\0&0&0&1\\0&0&0&\frac{3}{2}\\0&0&0&2\end{bmatrix}\ d=\begin{bmatrix}0&0&0&\frac{1539551}{8709120}\\0&0&0&\frac{24463}{136080}\\0&0&0&\frac{6501}{35840}\\0&0&0&\frac{1601}{8505}\end{bmatrix}\ c=\begin{bmatrix}0&0&0&\frac{26051}{2903040}\\0&0&0&\frac{421}{45360}\\0&0&0&\frac{339}{35840}\\0&0&0&\frac{29}{2835}\end{bmatrix}$$

$$b=\begin{bmatrix}\frac{89371}{544320}&\frac{103}{1260}&\frac{38341}{544320}&\frac{59681}{8709120}\\\frac{3308}{8505}&\frac{104}{315}&\frac{788}{8505}&\frac{1153}{136080}\\\frac{921}{2240}&\frac{81}{140}&\frac{711}{2240}&\frac{411}{35840}\\\frac{4096}{8505}&\frac{208}{315}&\frac{4096}{8505}&\frac{1601}{8505}\end{bmatrix}\ p=\begin{bmatrix}-\frac{31207}{362880}&-\frac{81}{1280}&-\frac{1243}{72576}&-\frac{2237}{2903040}\\-\frac{38}{567}&-\frac{1}{10}&-\frac{62}{2835}&-\frac{43}{45360}\\-\frac{279}{4480}&-\frac{81}{1280}&-\frac{183}{4480}&-\frac{9}{7168}\\-\frac{128}{2835}&0&\frac{128}{2835}&-\frac{29}{2835}\end{bmatrix}$$

Substituting this equation into method 3.3.2 yields the following stability polynomial

$$\left(-\frac{1}{2304}w^3+\frac{1}{806400}w^4\right)h^8+\left(-\frac{17113}{604800}w^3+\frac{103}{483840}w^4\right)h^6+\left(-\frac{230633}{453600}w^3+\frac{8599}{725760}w^4\right)h^4$$
$$+\left(-\frac{25279}{11240}w^3+\frac{1229}{5184}w^4\right)h^2+w^4-\frac{2864}{2835}w^3$$

This is obtained with Maple 18 software, hence using Mat lab 10 software we obtained the region of absolute stability as shown in figure 1 below.

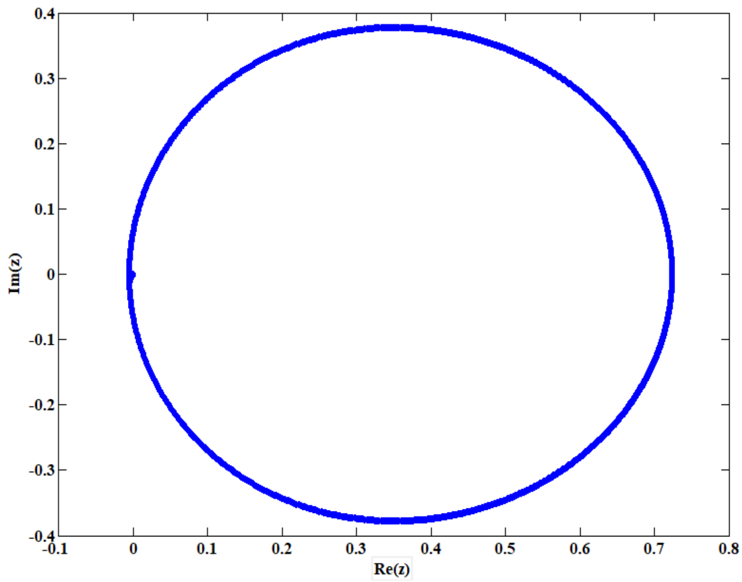


Figure 1: Region of Absolute Stability for K =2 with 2 off grid

4.6 Numerical Experiment:

Example 1: Consider the mildly stiffly differential equation

$$y_1^1 = -8y_1 + 7y_2; y_1(0) = 1$$
$$y_2^1 = 42y_1 - 43y_2; y_2(0) = 8, h = 0.1$$

with exact solution

$$y_1(x) = 2e^{-x} - e^{-50x}$$

$$y_2(x) = 2e^{-x} - 6e^{-50x}$$

Source: **(**Donald, J. Z., Skwame, Y. and Dedan, G. (2018): DSD)

NB
2S2OGP: Two step with Two Off-grid Point
DSD: Donald, J. Z., Skwame, Y. and Dedan, G. (2018).

Example 2: Consider the mildly stiffly differential equation

$$y_1^1 = -y_1 + 95y_2; y_1(0) = 1$$

$$y_2^1 = -y_1 - 97y_2; y_2(0) = 8, h = 0.1$$

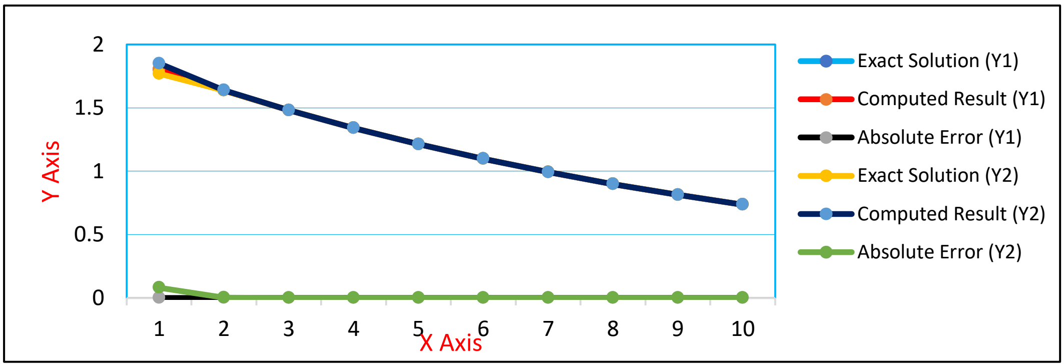
with exact solution

$$y_1(x) = \frac{95}{47}e^{-2x} - \frac{48}{47}e^{-96x}$$

$$y_2(x) = \frac{48}{47}e^{-96x} - \frac{1}{47}e^{-2x}$$

Table 1: The Exact Solution and the Computed Result for Experiment 1 of 2S2OGP

X	Exact Solution (y_1)	Computed Result (y_1)	Absolute Error (y_1)	Exact Solution (y_2)	Computed Result (y_2)	Absolute Error (y_2)
0.1	1.8029368819	1.8028987080	0.0000381739	1.7692471540	1.8503316120	0.0810844580
0.2	1.6374161060	1.6371934530	0.0002226530	1.6371891060	1.6390698190	0.0018807130
0.3	1.4816361350	1.4816346250	0.0000015100	1.4816346060	1.4816473380	0.0000127320
0.4	1.3406400900	1.3406400200	0.0000000700	1.3406400800	1.3406405200	0.0000004400
0.5	1.2130613190	1.2130613200	0.0000000010	1.2130613190	1.2130613220	0.0000000030
0.6	1.0976232720	1.0976232720	0.0000000000	1.0976232720	1.0976232720	0.0000000000
0.7	0.9931706096	0.9931706079	0.0000000017	0.9931706076	0.9931706075	0.0000000001
0.8	0.8986579282	0.8986579284	0.0000000002	0.8986579284	0.8986579267	0.0000000017
0.9	0.8131393194	0.8131393198	0.0000000004	0.8131393194	0.8131393193	0.0000000001
1.0	0.7357588824	0.7357588821	0.0000000003	0.7357588824	0.7357588811	0.0000000013



	Example 1		Example 2		DSD (2018)	
x	**Absolute Error (y_1)**	**Absolute Error (y_2)**	**Absolute Error (y_1)**	**Absolute Error (y_2)**	**Absolute Error (y_1)**	**Absolute Error (y_2)**
0.1	7.73E-04	1.95E-02	7.73 E-04	1.95 E-02	6.31E-01	6.07E-01
0.2	7.74E-03	5.72E-03	7.74 E-04	5.71 E-03	3.72E-01	3.90E-01
0.3	7.29E-07	1.18E-02	7.29 E-07	1.18 E-03	1.75E-01	1.85E-01
0.4	5.86E-05	9.67E-03	5.86 E-07	9.67E-03	1.05E-01	1.10E-01
0.5	4.65E-08	8.00E-03	4.65 E-08	8.01 E-03	7.39E-02	7.07E-02
0.6	4.42E-08	6.41 E-03	4.42 E-08	6.+41E-03	4.43E-02	3.36E-02
0.7	1.60E-09	5.24 E-03	1.6 E-09	5.24 E-03	1.82E-02	1.99E-02
0.8	1.20E-09	4.29 E-03	1.2 E-0.9	4.29 E-03	1.46E-02	1.28E-02
0.9	1.80E-09	3.51 E-03	1.8 E-09	3.51 E-03	2.80E-03	5.96E-03
1.0	2.00E-10	2.87E-03	2.0 E-09	2.878E-03	6.94E-03	3.63E-03

Figure 2: showing the performance of our methods with exact solution and absolute error of examples 1 and 2

Table 2: The Exact Solution and the Computed Result for Example 2 of 2S2OGP

X	Exact Solution (y_1)	Computed Result (y_1)	Absolute Error (y_1)	Exact Solution (y_2)	Computed Result (y_2)	Absolute Error (y_2)
0.1	1.654812139	1.654038801	0.000773338	0.00300963809	-0.0165772959	0.019586934
0.2	1.354902216	1.347162225	0.007739991	0.00028509549	-0.0054221324	0.005707228
0.3	1.109293001	1.109293731	0.000000729	0.00020402520	-0.0116704582	0.011874483
0.5	0.743586104	0.743586058	0.000000046	0.00017563244	-0.0078271737	0.008002806
0.6	0.608796811	0.608796369	0.000000442	0.00000358383	-0.0064079429	0.006411526
0.7	0.498440671	0.498440673	0.000000001	-0.00000333119	-0.0052467435	0.005243412
0.8	0.408088706	0.408088705	0.000000001	-0.00000133167	-0.0042956672	0.004294335
0.9	0.334114774	0.334114775	0.000000001	-0.00000117702	-0.0035169976	0.003515820
1.0	0.273550040	0.273550042	0.000000002	-0.00000096178	-0.0028794741	0.002878512

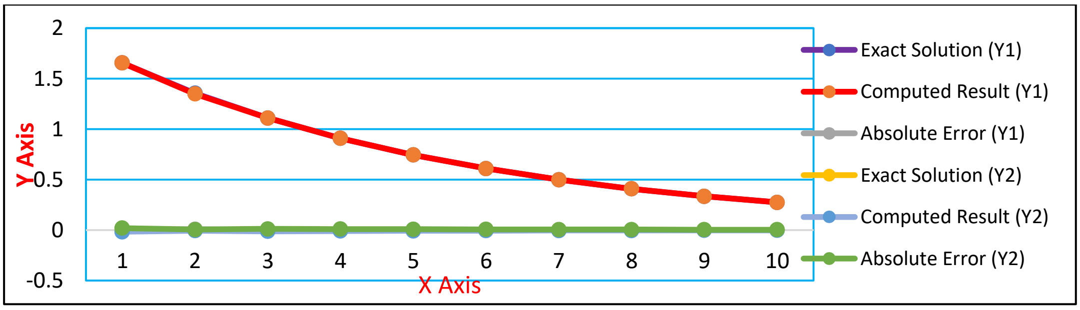


Figure 3: showing the performance of our methods with exact solution and absolute error of experiment 2

Table 3: Comparison of new methods with that of Donald, J. Z., Skwame, Y. and Dedan, G. (2018): DSD on examples 1 and 2.

V. CONCLUSIONS

The numerical results obtained in table 3 for the problems solved suggest that the new proposed block hybrid method (3.11) is suitable for stiff problems and perform competitively well with less computational effort compared with the method of Donald et al (2018).

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