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An A-Stable Backward Difference Second Order Linear Multistep Method for Solving Stiff Ordinary Differential Equation

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ABSTRACT: An A-stable backward difference second derivative linear multistep method for solving stiff ordinary differential equation via multi-step interpolation and collocation methods has be studied in this research. In deriving the methods, we use the power series as a basis function to obtain the main continuous and discrete. The analysis of the basic properties of the new methods has been studied in this research, and it was found to be consistent, zero-stable, with uniformly order twelve. We further plotted the region of absolute stability of our methods which the method is A-stable. The newly constructed methods was applied to solve the systems stiff initial value problems of general of second-order ordinary differential equations and the results was compared with the existing methods. It is evident that, our method performs better than the existing methods.

KEYWORD: A-stable, Backward Difference, Linear Multistep, Stiff ODEs.

I. INTRODUCTION

Most real-life problems that arise in various fields of study be it engineering or science are modeled as mathematical models before they are solved. These models often lead to differential equations. A differential equation can simply be defined as an equation that contains derivative(s). In other words, it’s a relationship involving an independent variable, a dependent variable and one or more differential co-efficient of with respect to (Fatunla, 1988). This research focuses on solving second order initial value problems of ordinary differential equations (ODEs) of the form

$$y'' = f\left(x, y, y'\right); \quad y\left(x_0\right) = a, \quad y'\left(x_0\right) = b, \quad a \le x \le b \qquad (1.1)$$

The development of algorithms has been largely guided by convergence theorems of (Dahlquist, 1956, 1959, 1963 and 1978) as well as the treatises of (Henrici, 1962), (Stetter, 1973) and (Fatunla, 1988). An ordinary differential equation is one for which the unknown function (also known as dependent variable) is a function of a single independent variable. An ODE is classified according to the order of the highest derivative with respect to the dependent variable, (Skwame, Sunday and Sabo, 2018). Many researchers have proposed various forms of linear multi-step methods for the solutions of stiff ordinary differential equations. Most of these improvements in the class of linear multistep methods have been based on backward differentiation formula (BDF), because of its special properties. Among the first modifications introduced by different authors was the Extended Backward Differentiation formulas (EBDFs), introduced in 1980 by (Cash, 1981) in which one-super future point technique was applied. (Cash, 1980) developed the integration of stiff systems of ODEs using extended backward differentiation formula in (1981), Cash, formed Second derivative extended backward differentiation formulas for the numerical integration of stiff systems and the integration of stiff initial value problems in ODEs using modified extended backward differentiation formula. The first use of BDF methods appears to date back to (Curtiss and Hirschfelder, 1952), although then they were not given that name.

Curtiss and Hirschfelder said stiff equations are equations where certain implicit methods, and in particular backward differentiation formulas (BDFs), perform better, usually tremendously better, than explicit ones. The importance of the BDF-based methods is their stability: they are stable along the entire negative real axis. The study of Second derivative Hybrid Block Backward Differentiation formula for Numerical Solution of Stiff Systems was carried out by (Skwame, Kumleng and Bakari, 2017). The authors, who engaged in forming block methods, include: (Alkasassbeh and Zurni, 2017), (Ibijola, Skwame and Kumleng, 2011), (Skwame, Sabo, Tumba and Kyagya, 2017), (Skwame, Sunday and Sabo, 2018), (Raf’t and Zurni, 2016), (Omar and Alkasassebeh, 2016).

II. CONSTRUCTION OF METHODS

In this section, An A-stable backward difference second derivative linear multistep method equation for the solution of stiff second order ordinary differential equations of the form (1.1). Let the power series of the form

$$y(x)=\sum_{j=0}^{r+s-1} a_j\left(\frac{x-x_n}{h}\right)^j \tag{2.1}$$

be the approximate solution to equation (1.1) for where is the sum of the number of collocation and interpolation points, are the real coefficients to be determined, is a constant step size of the partition of interval , which is given, by the partition . The method will be derived by the introduction of off-mesh following the method of (Mohammed and Raphael, 2015), (Nwachukwu and Okor, 2018), (Skwame, Sunday and Sabo, 2018), (Abdelrahhim and Omar, 2016) and currently (Cash, 1981, 1983) and (Mohammed and Omar, 2017). Differentiating equation (2.1) once and twice yield

$$^j y'(x)={}^j f(x^j, y^j, {}^j y)=\sum_{i=0}^{r+s-1}\frac{ia_i}{h}\left(\frac{x-x_n}{h}\right)^{i-1}, \quad j=1,\cdots,m. \tag{2.2}$$

$$^j y''(x)={}^j f(x^j, y^j, {}^j y')=\sum_{i=0}^{r+s-1}\frac{i(i-1)a_i}{h^2}\left(\frac{x-x_n}{h}\right)^{i-2}, \quad j=1,\cdots,m. \tag{2.3}$$

interpolating equation (2.1) at and collocating (2.3) at where and represent the number of collocation, interpolation and off-step points respectively and is the step number, leads to the following system of equations

$$^j y''(x)={}^j f(x^j, y^j, {}^j y')=\sum_{i=0}^{r+s-1}\frac{i(i-1)a_i}{h^2}\left(\frac{x-x_n}{h}\right)^{i-2}, g_{n+r}, r=0, v_i, k, j=1,2,\cdots,m. \tag{2.4}$$

$$y(x)=\sum_{j=0}^{r+s-1}\frac{1}{h}a_j(x-x_{n+s})^j=y_{n+s}, s=0, v_i, i, j=1,2,\cdots,m \tag{2.5}$$

To derive this method, three-step with two off-mesh points, for this method, the off-mesh point is we have a polynomial of degree as follows:

$$y(x)=\sum_{j=0}^{12} a_j\left(\frac{x-x_n}{h}\right)^j \tag{2.6}$$

with first and second derivative given by

$$y'(x)=\sum_{j=0}^{12} ja_j\left(\frac{x-x_n}{h}\right)^{j-1} \tag{2.7}$$

$$y''(x)=\sum_{j=0}^{12} j(j-1)a_j\left(\frac{x-x_n}{h}\right)^{j-2} \tag{2.8}$$

Now interpolating (3.18) at and collocating (2.9) and (2.10) at leads to a system of equations written in the matrix form as

$$
\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} & x_n^{11} & x_n^{12} \\
0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 & 10x_n^9 & 11x_n^{10} & 12x_n^{11} \\
0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 & 8x_{n+1}^7 & 9x_{n+1}^8 & 10x_{n+1}^9 & 11x_{n+1}^{10} & 12x_{n+1}^{11} \\
0 & 1 & 2x_{n+\frac{3}{2}} & 3x_{n+\frac{3}{2}}^2 & 4x_{n+\frac{3}{2}}^3 & 5x_{n+\frac{3}{2}}^4 & 6x_{n+\frac{3}{2}}^5 & 7x_{n+\frac{3}{2}}^6 & 8x_{n+\frac{3}{2}}^7 & 9x_{n+\frac{3}{2}}^8 & 10x_{n+\frac{3}{2}}^9 & 11x_{n+\frac{3}{2}}^{10} & 12x_{n+\frac{3}{2}}^{11} \\
0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 & 8x_{n+2}^7 & 9x_{n+2}^8 & 10x_{n+2}^9 & 11x_{n+2}^{10} & 12x_{n+2}^{11} \\
0 & 1 & 2x_{n+\frac{5}{2}} & 3x_{n+\frac{5}{2}}^2 & 4x_{n+\frac{5}{2}}^3 & 5x_{n+\frac{5}{2}}^4 & 6x_{n+\frac{5}{2}}^5 & 7x_{n+\frac{5}{2}}^6 & 8x_{n+\frac{5}{2}}^7 & 9x_{n+\frac{5}{2}}^8 & 10x_{n+\frac{5}{2}}^9 & 11x_{n+\frac{5}{2}}^{10} & 12x_{n+\frac{5}{2}}^{11} \\
0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 & 8x_{n+3}^7 & 9x_{n+3}^8 & 10x_{n+3}^9 & 11x_{n+3}^{10} & 12x_{n+3}^{11} \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 & 90x_n^8 & 110x_n^9 & 132x_n^{10} \\
0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 & 72x_{n+1}^7 & 90x_{n+1}^8 & 110x_{n+1}^9 & 132x_{n+1}^{10} \\
0 & 0 & 2 & 6x_{n+\frac{3}{2}} & 12x_{n+\frac{3}{2}}^2 & 20x_{n+\frac{3}{2}}^3 & 30x_{n+\frac{3}{2}}^4 & 42x_{n+\frac{3}{2}}^5 & 56x_{n+\frac{3}{2}}^6 & 72x_{n+\frac{3}{2}}^7 & 90x_{n+\frac{3}{2}}^8 & 110x_{n+\frac{3}{2}}^9 & 132x_{n+\frac{3}{2}}^{10} \\
0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 & 72x_{n+2}^7 & 90x_{n+2}^8 & 110x_{n+2}^9 & 132x_{n+2}^{10} \\
0 & 0 & 2 & 6x_{n+\frac{5}{2}} & 12x_{n+\frac{5}{2}}^2 & 20x_{n+\frac{5}{2}}^3 & 30x_{n+\frac{5}{2}}^4 & 42x_{n+\frac{5}{2}}^5 & 56x_{n+\frac{5}{2}}^6 & 72x_{n+\frac{5}{2}}^7 & 90x_{n+\frac{5}{2}}^8 & 110x_{n+\frac{5}{2}}^9 & 132x_{n+\frac{5}{2}}^{10} \\
0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 & 72x_{n+3}^7 & 90x_{n+3}^8 & 110x_{n+3}^9 & 132x_{n+3}^{10}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \end{bmatrix}
=
\begin{bmatrix} y_n \\ f_n \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ g_n \\ g_{n+1} \\ g_{n+\frac{3}{2}} \\ g_{n+2} \\ g_{n+\frac{5}{2}} \\ g_{n+3} \end{bmatrix}
\qquad (211)
$$

Using Gaussian elimination method, on (2.11) gives the coefficient . The values are then substituted into equation (2.1) to give the implicit continuous hybrid method of the form;

$$
p(x)=\sum_{i=0}\alpha_i y_{n+i}+h\left[\sum_{i=0,1,\frac{3}{2},2,\frac{5}{2},3}\beta_i f_{n+i}+\sum_{i=0}^{3}\beta_i f_{n+i}\right]+h^2\left[\sum_{i=0,1,\frac{3}{2},2,\frac{5}{2},3}\gamma_i g_{n+i}+\sum_{i=0}^{3}\gamma_i g_{n+i}\right] \qquad (2.12)
$$

differentiating equation (2.12) once

$$
p'(x)=\frac{1}{h}\sum_{i=0}\alpha_i y_{n+i}+\left[\sum_{i=0,1,\frac{3}{2},2,\frac{5}{2},3}\beta_i f_{n+i}+\sum_{i=0}^{3}\beta_i f_{n+i}\right]+h\left[\sum_{i=0,1,\frac{3}{2},2,\frac{5}{2},3}\gamma_i g_{n+i}+\sum_{i=0}^{3}\gamma_i g_{n+i}\right] \qquad (2.13)
$$

where

$$\alpha_0 = 0$$

$$\beta_0 = -x_n + x - \frac{16907}{2700}\frac{(-x_n+x)^3}{h^2} + \frac{288101}{18000}\frac{(-x_n+x)^4}{h^3} - \frac{433807}{20250}\frac{(-x_n+x)^5}{h^4}$$

$$+ \frac{443659}{24300}\frac{(-x_n+x)^6}{h^5}$$

$$- \frac{28447}{2700}\frac{(-x_n+x)^7}{h^6} + \frac{452159}{108000}\frac{(-x_n+x)^8}{h^7} - \frac{6878}{6075}\frac{(-x_n+x)^9}{h^8} + \frac{673}{3375}\frac{(-x_n+x)^{10}}{h^9}$$

$$- \frac{92}{4455}\frac{(-x_n+x)^{11}}{h^{10}} + \frac{29}{30375}\frac{(-x_n+x)^{12}}{h^{11}}$$

$$\beta_1 = -\frac{400(-x_n+x)^3}{h^2} + \frac{5985}{4}\frac{(-x_n+x)^4}{h^3} - \frac{38761}{15}\frac{(-x_n+x)^5}{h^4} + \frac{190823}{72}\frac{(-x_n+x)^6}{h^5}$$

$$- \frac{95105}{54}\frac{(-x_n+x)^7}{h^6} + \frac{675643}{864}\frac{(-x_n+x)^8}{h^7} - \frac{56188}{243}\frac{(-x_n+x)^9}{h^8} + \frac{1183}{27}\frac{(-x_n+x)^{10}}{h^9}$$

$$- \frac{1432}{297}\frac{(-x_n+x)^{11}}{h^{10}} + \frac{19}{81}\frac{(-x_n+x)^{12}}{h^{11}}$$

$$\beta_{\frac{3}{2}} = -\frac{12800}{27}\frac{(-x_n+x)^3}{h^2} + \frac{52480}{27}\frac{(-x_n+x)^4}{h^3} - \frac{1488512}{405}\frac{(-x_n+x)^5}{h^4} + \frac{110912}{27}\frac{(-x_n+x)^6}{h^5}$$

$$- \frac{239360}{81}\frac{(-x_n+x)^7}{h^6} + \frac{114208}{81}\frac{(-x_n+x)^8}{h^7} - \frac{324224}{729}\frac{(-x_n+x)^9}{h^8} + \frac{7232}{81}\frac{(-x_n+x)^{10}}{h^9}$$

$$- \frac{1024}{99}\frac{(-x_n+x)^{11}}{h^{10}} + \frac{128}{243}\frac{(-x_n+x)^{12}}{h^{11}}$$

$$\beta_2 = \frac{2025}{4}\frac{(-x_n+x)^3}{h^2} - \frac{31185}{16}\frac{(-x_n+x)^4}{h^3} + \frac{34641}{10}\frac{(-x_n+x)^5}{h^4} - \frac{14577}{4}\frac{(-x_n+x)^6}{h^5}$$

$$+ \frac{9885}{4}\frac{(-x_n+x)^7}{h^6} - \frac{35617}{32}\frac{(-x_n+x)^8}{h^7} + \frac{2986}{9}\frac{(-x_n+x)^9}{h^8} - \frac{63(-x_n+x)^{10}}{h^9}$$

$$+ \frac{76}{11}\frac{(-x_n+x)^{11}}{h^{10}} - \frac{1}{3}\frac{(-x_n+x)^{12}}{h^{11}}$$

$$\beta_{\frac{5}{2}} = \frac{8704}{25}\frac{(-x_n+x)^3}{h^2} - \frac{175104}{125}\frac{(-x_n+x)^4}{h^3} + \frac{980608}{375}\frac{(-x_n+x)^5}{h^4} - \frac{653248}{225}\frac{(-x_n+x)^6}{h^5}$$

$$+ \frac{1408768}{675}\frac{(-x_n+x)^7}{h^6} - \frac{3375712}{3375}\frac{(-x_n+x)^8}{h^7} + \frac{1932928}{6075}\frac{(-x_n+x)^9}{h^8}$$

$$- \frac{218048}{3375}\frac{(-x_n+x)^{10}}{h^9} + \frac{1024}{135}\frac{(-x_n+x)^{11}}{h^{10}} - \frac{3968}{10125}\frac{(-x_n+x)^{12}}{h^{11}}$$

$$\beta_3 = \frac{700}{27}\frac{(-x_n+x)^3}{h^2} - \frac{11455}{108}\frac{(-x_n+x)^4}{h^3} + \frac{81718}{405}\frac{(-x_n+x)^5}{h^4} - \frac{444893}{1944}\frac{(-x_n+x)^6}{h^5}$$

$$+ \frac{27295}{162}\frac{(-x_n+x)^7}{h^6} - \frac{214913}{2592}\frac{(-x_n+x)^8}{h^7} + \frac{19796}{729}\frac{(-x_n+x)^9}{h^8} - \frac{461}{81}\frac{(-x_n+x)^{10}}{h^9}$$

$$+ \frac{56}{81}\frac{(-x_n+x)^{11}}{h^{10}} - \frac{1}{27}\frac{(-x_n+x)^{12}}{h^{11}}$$

$$\gamma_0 = \frac{1}{2}\left(x-x_n\right)^2 - \frac{29}{15}\frac{\left(x-x_n\right)^3}{h} + \frac{13369}{3600}\frac{\left(x-x_n\right)^4}{h^2} - \frac{332}{75}\frac{\left(x-x_n\right)^5}{h^3} + \frac{17221}{4860}\frac{\left(x-x_n\right)^6}{h^4} - \frac{1115}{567}\frac{\left(x-x_n\right)^7}{h^5} + \frac{49313}{64800}\frac{\left(x-x_n\right)^8}{h^6} - \frac{736}{3645}\frac{\left(x-x_n\right)^9}{h^7} + \frac{71}{2025}\frac{\left(x-x_n\right)^{10}}{h^8}$$

$$\gamma_1 = -\frac{75\left(x-x_n\right)^3}{h} + \frac{270\left(x-x_n\right)^4}{h^2} - \frac{9049}{20}\frac{\left(x-x_n\right)^5}{h^3} + \frac{10871}{24}\frac{\left(x-x_n\right)^6}{h^4} - \frac{74371}{252}\frac{\left(x-x_n\right)^7}{h^5} + \frac{37129}{288}\frac{\left(x-x_n\right)^8}{h^6} - \frac{3046}{81}\frac{\left(x-x_n\right)^9}{h^7} + \frac{317}{45}\frac{\left(x-x_n\right)^{10}}{h^8} - \frac{76}{99}\frac{\left(x-x_n\right)^{11}}{h^9} + \frac{1}{27}\frac{\left(x-x_n\right)^{12}}{h^{10}}$$

$$\gamma_{\frac{3}{2}} = -\frac{3200}{9}\frac{\left(x-x_n\right)^3}{h} + \frac{12320}{9}\frac{\left(x-x_n\right)^4}{h^2} - \frac{329248}{135}\frac{\left(x-x_n\right)^5}{h^3} + \frac{626912}{243}\frac{\left(x-x_n\right)^6}{h^4} - \frac{1001024}{567}\frac{\left(x-x_n\right)^7}{h^5} + \frac{65248}{81}\frac{\left(x-x_n\right)^8}{h^6} - \frac{178016}{729}\frac{\left(x-x_n\right)^9}{h^7} + \frac{19168}{405}\frac{\left(x-x_n\right)^{10}}{h^8} - \frac{4736}{891}\frac{\left(x-x_n\right)^{11}}{h^9} + \frac{64}{243}\frac{\left(x-x_n\right)^{12}}{h^{10}}$$

$$\gamma_2 = -\frac{675}{2}\frac{\left(x-x_n\right)^3}{h} + \frac{21465}{16}\frac{\left(x-x_n\right)^4}{h^2} - \frac{12357}{5}\frac{\left(x-x_n\right)^5}{h^3} + \frac{10821}{4}\frac{\left(x-x_n\right)^6}{h^4} - \frac{26821}{14}\frac{\left(x-x_n\right)^7}{h^5} + \frac{28929}{32}\frac{\left(x-x_n\right)^8}{h^6} - \frac{2548}{9}\frac{\left(x-x_n\right)^9}{h^7} + \frac{283}{5}\frac{\left(x-x_n\right)^{10}}{h^8} - \frac{72}{11}\frac{\left(x-x_n\right)^{11}}{h^9} + \frac{1}{3}\frac{\left(x-x_n\right)^{12}}{h^{10}}$$

$$\gamma_{\frac{5}{2}} = -\frac{384}{5}\frac{\left(x-x_n\right)^3}{h} + \frac{7776}{25}\frac{\left(x-x_n\right)^4}{h^2} - \frac{14624}{25}\frac{\left(x-x_n\right)^5}{h^3} + \frac{9824}{15}\frac{\left(x-x_n\right)^6}{h^4} - \frac{149696}{315}\frac{\left(x-x_n\right)^7}{h^5} + \frac{51776}{225}\frac{\left(x-x_n\right)^8}{h^6} - \frac{29984}{405}\frac{\left(x-x_n\right)^9}{h^7} + \frac{3424}{225}\frac{\left(x-x_n\right)^{10}}{h^8} - \frac{896}{495}\frac{\left(x-x_n\right)^{11}}{h^9} + \frac{64}{675}\frac{\left(x-x_n\right)^{12}}{h^{10}}$$

$$\gamma_3 = -\frac{25}{9}\frac{\left(x-x_n\right)^3}{h} + \frac{205}{18}\frac{\left(x-x_n\right)^4}{h^2} - \frac{11729}{540}\frac{\left(x-x_n\right)^5}{h^3} + \frac{48031}{1944}\frac{\left(x-x_n\right)^6}{h^4} - \frac{41393}{2268}\frac{\left(x-x_n\right)^7}{h^5} + \frac{23369}{2592}\frac{\left(x-x_n\right)^8}{h^6} - \frac{2162}{729}\frac{\left(x-x_n\right)^9}{h^7} + \frac{253}{405}\frac{\left(x-x_n\right)^{10}}{h^8} - \frac{68}{891}\frac{\left(x-x_n\right)^{11}}{h^9} + \frac{1}{243}\frac{\left(x-x_n\right)^{12}}{h^{10}}$$

On evaluating (2.13) at $x = x_{n+i},\ i = 0, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ yield the discrete shames

$$y_{n+1} = y_n + \frac{987317}{3564000} h f_n - \frac{3723047}{427680} h f_{n+1} - \frac{329888}{40095} h f_{n+\frac{3}{2}} + \frac{168059}{15840} h f_{n+2} + \frac{199712}{30375} h f_{n+\frac{5}{2}} + \frac{54959}{116640} h f_{n+3} + \frac{954439}{44906400} h^2 g_n - \frac{1981547}{997920} h^2 g_{n+1} - \frac{2109248}{280665} h^2 g_{n+\frac{3}{2}} - \frac{727949}{110880} h^2 g_{n+2} - \frac{222752}{155925} h^2 g_{n+\frac{5}{2}} - \frac{90239}{1796256} h^2 g_{n+3}$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3120961}{11264000} h f_n - \frac{3833271}{450560} h f_{n+1} - \frac{14081}{1760} h f_{n+\frac{3}{2}} + \frac{4804839}{450560} h f_{n+2} + \frac{26379}{4000} h f_{n+\frac{5}{2}} + \frac{19349}{40960} h f_{n+3} + \frac{335283}{15769600} h^2 g_n - \frac{6224769}{3153920} h^2 g_{n+1} - \frac{186639}{24640} h^2 g_{n+\frac{3}{2}} - \frac{4156029}{630784} h^2 g_{n+2} - \frac{176499}{123200} h^2 g_{n+\frac{5}{2}} - \frac{158841}{3153920} h^2 g_{n+3}$$

$$y_{n+2} = y_n + \frac{92581}{334125} h f_n - \frac{113612}{13365} h f_{n+1} - \frac{311296}{40095} h f_{n+\frac{3}{2}} + \frac{5399}{495} h f_{n+2} + \frac{200704}{30375} h f_{n+\frac{5}{2}} + \frac{1724}{3645} h f_{n+3} + \frac{29839}{1403325} h^2 g_n - \frac{12304}{6237} h^2 g_{n+1} - \frac{2119168}{280665} h^2 g_{n+\frac{3}{2}} - \frac{22931}{3465} h^2 g_{n+2} - \frac{223744}{155925} h^2 g_{n+\frac{5}{2}} - \frac{14152}{280665} h^2 g_{n+3}$$

$$y_{n+\frac{5}{2}} = y_n + \frac{2022565}{7299072} h f_n - \frac{186000625}{21897216} h f_{n+1} - \frac{1983125}{256608} h f_{n+\frac{3}{2}} + \frac{9041875}{811008} h f_{n+2} + \frac{53045}{7776} h f_{n+\frac{5}{2}} + \frac{2835625}{5971968} h f_{n+3} + \frac{9778675}{459841536} h^2 g_n - \frac{100748125}{51093504} h^2 g_{n+1} - \frac{27086875}{3592512} h^2 g_{n+\frac{3}{2}} - \frac{37313125}{5677056} h^2 g_{n+2} - \frac{579275}{399168} h^2 g_{n+\frac{5}{2}} - \frac{23268125}{459841536} h^2 g_{n+3}$$

$$y_{n+3} = y_n + \frac{12197}{44000} h f_n - \frac{14877}{1760} h f_{n+1} - \frac{416}{55} h f_{n+\frac{3}{2}} + \frac{19683}{1760} h f_{n+2} + \frac{864}{125} h f_{n+\frac{5}{2}} + \frac{103}{160} h f_{n+3} + \frac{1311}{61600} h^2 g_n - \frac{24219}{12320} h^2 g_{n+1} - \frac{576}{77} h^2 g_{n+\frac{3}{2}} - \frac{79461}{12320} h^2 g_{n+2} - \frac{2592}{1925} h^2 g_{n+\frac{5}{2}} - \frac{723}{12320} h^2 g_{n+3}$$

III. ANALYSIS OF THE METHOD

In this section, the analysis of the basic properties of the new method which include order, error constant, consistency, zero stability and the region of absolute stability derived shall be analyzed.

Order and error constant of the method (Definition 3.1): Following (Butcher, 2009) the linear difference operator associated with the LMM is defined by

$$\ell[y(x),h] = \sum_{j=0}^{k}\left[(a_j y(x+jh) - h\beta_j y'(x+jh)\right] \tag{3.1}$$

applying Taylor expansion about the point x, we get

$$\ell\,[y(x),h] = c_0 y(x) + c_1 h y^{/}(x) + \cdots + c_q h^q y^q(x) + \cdots \tag{3.2}$$

Expand (3.2) in Taylor series about x, Comparing the coefficient of h gives

$C_0 = C_1 = C_2 = C_3 = \ldots = C_p = 0$ and

$C_{13} = \left[5.6232\times10^{-9},\quad 5.6327\times10^{-9},\quad 5.6359\times10^{-9},\quad 5.6412\times10^{-9},\quad 5.6920\times10^{-9}\right]^T$

Consistency of the method: According to (Lambert, 1973), the hybrid block method) is said to be consistent if it has an order more than or equal to one. Therefore, our method is consistent, since the order is more than one.

Zero Stability of our Method According to (Lambert, 1973), a block method is said to be zero-stable if as, the root of the

first characteristic polynomial $\rho(z)=0$ that is $\rho(z)=\det\left[\sum_{i=0}^{k} A^{(i)} z^{k-i}\right]=0$ Satisfies $|z_i|\le 1$ and for those roots

with $|z_i|=1$, multiplicity must not exceed two. The block method expressed in the form

$$\rho(z)=\left| z\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = z^4(z-1)$$

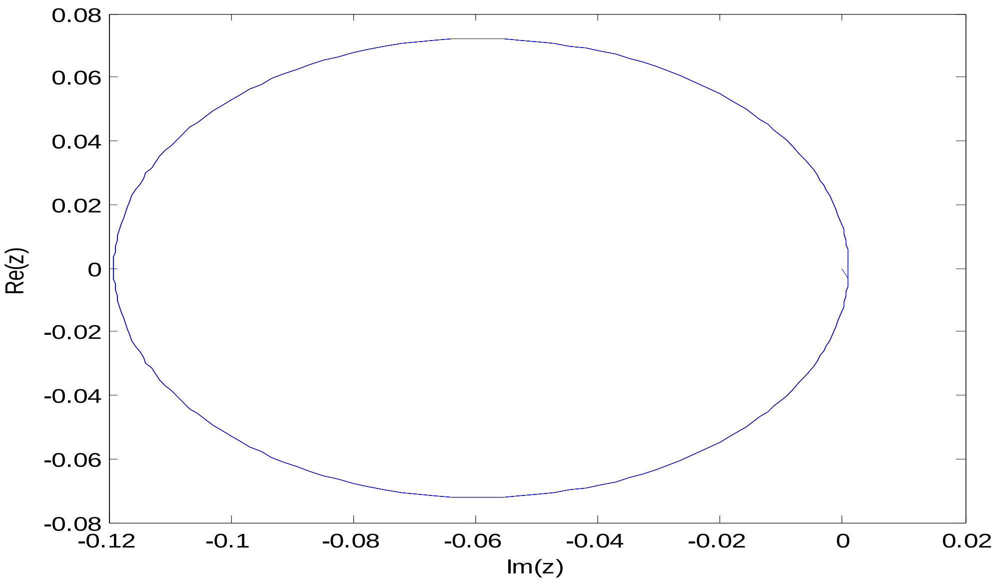
$$\rho(z)=z^4(z-1)=0, \quad z=0,0,0,0,1$$

Hence, our method is zero-stable.

Convergence: Convergence is an essential property that every acceptable linear multistep method (LMM) must possess. According to Dahlquist, (1963), consistency and zero stability are the necessary conditions for the convergence of any numerical method.

Theorem (3.1): The consistency and zero stability are sufficient condition for linear multistep method to be convergent. Since the hybrid block method is consistent and zero stable, it implies that the method is convergent for all point.

Region of Absolute Stability of: According to (Dahlquist, 1956 and Lambert, 1991), the absolute stability region of the new method is plotted using Mat Lab software and is shown in the figure below.



IV. NUMERICAL EXPERIMENTS

In order to study the efficiency of the developed methods, we present three numerical experiments widely solved by (Skwame, Kumleng and Bakari, 2017) and (Althemai, Skwame and Donald, 2014).

Experiment 5.1 Consider the stiffly differential equation

$$y_1' = 198y_1 + 199y_2; \quad y_1(0) = 1$$
$$y_2' = -398y_1 - 399y_2; \quad y_2(0) = -1, \quad h = 0.1$$

with Exact Solution

$$y_1(x) = e^{-x}$$
$$y_2(x) = -e^{-x}$$

Source: (Skwame, Kumleng and Bakari, 2017b).

Table 5.1: Comparison of new method with that of (Skwame, Kumleng and Bakari, 2017b)

X	Errors in our method		Error in (Skwame, Kumleng and Bakari, 2017b)	
	$y(x_1)$	$y(x_2)$	$y(x_1)$	$y(x_2)$
0.1	3.72×10^{-6}	3.50×10^{-6}	3.60×10^{-6}	3.60×10^{-6}
0.2	2.90×10^{-7}	3.53×10^{-7}	2.42×10^{-6}	2.42×10^{-6}
0.3	1.87×10^{-6}	1.82×10^{-6}	3.18×10^{-6}	3.18×10^{-6}
0.4	4.50×10^{-6}	4.33×10^{-6}	3.90×10^{-6}	3.90×10^{-6}
0.5	1.79×10^{-6}	1.84×10^{-6}	3.58×10^{-6}	3.58×10^{-6}
0.6	2.82×10^{-6}	2.78×10^{-6}	3.23×10^{-6}	3.23×10^{-6}
0.7	4.63×10^{-6}	4.51×10^{-6}	4.35×10^{-6}	4.35×10^{-6}
0.8	2.50×10^{-6}	2.54×10^{-6}	3.97×10^{-6}	3.97×10^{-6}
0.9	3.15×10^{-6}	3.12×10^{-6}	3.59×10^{-6}	3.59×10^{-6}
1.0	4.39×10^{-6}	4.30×10^{-6}	4.31×10^{-6}	4.30×10^{-6}

Experiment 5.2
Consider the stiffly differential equation

$$y_1' = 998y_1 + 1998y_2; \quad y_1(0) = 1$$
$$y_2' = -999y_1 - 1999y_2; \quad y_2(0) = 0, \quad h = 0.1$$

with Exact Solution

$$y_1(x) = 2e^{-x} - e^{-1000x}$$
$$y_2(x) = -e^{-x} - e^{-1000x}$$

Source: (Skwame, Kumleng and Bakari, 2017b).

Table 5.1: Comparison of new method with that of (Skwame, Kumleng and Bakari, 2017b)

X	Errors in our method		Error in (Skwame, Kumleng and Bakari, 2017b)	
	$y(x_1)$	$y(x_2)$	$y(x_1)$	$y(x_2)$
0.1	4.51×10^{-3}	4.50×10^{-3}	2.43×10^{-2}	2.43×10^{-2}
0.2	3.45×10^{-4}	3.71×10^{-4}	3.87×10^{-2}	3.81×10^{-2}
0.3	2.57×10^{-2}	2.57×10^{-2}	9.31×10^{-4}	9.85×10^{-4}
0.4	1.20×10^{-4}	1.18×10^{-4}	1.51×10^{-5}	1.51×10^{-3}
0.5	3.38×10^{-5}	1.19×10^{-5}	2.32×10^{-5}	2.20×10^{-5}
0.6	6.51×10^{-4}	6.56×10^{-4}	6.99×10^{-5}	7.14×10^{-5}
0.7	1.70×10^{-6}	2.18×10^{-6}	2.15×10^{-5}	1.22×10^{-5}
0.8	3.65×10^{-5}	1.81×10^{-5}	2.34×10^{-5}	1.46×10^{-5}
0.9	5.02×10^{-6}	1.05×10^{-6}	2.17×10^{-5}	1.60×10^{-5}
1.0	4.21×10^{-6}	2.19×10^{-6}	1.97×10^{-5}	1.48×10^{-5}

Experiment 5.3
Consider the stiffly differential equation

$$y_1^1 = -8y_1 + 7y_2 \,;\, y_1(0) = 1,$$

$$y_2^1 = 42y_1 - 43y_2 \,;\, y_2(0) = 8, \quad h = \frac{1}{10} \,.$$

with Exact Solution

$$y_1(x) = 2e^{-x} - e^{-50x}$$

$$y_2(x) = 2e^{-x} - 6e^{-50x}$$

Source: (Althemai, Skwame and Donald, 2014).

Table 5.3: Comparison of new method with that of (Althemai, Skwame and Donald, 2014)

X	Errors in our method		Error in (Althemai, Skwame and Donald, 2014)	
	$y(x_1)$	$y(x_2)$	$y(x_1)$	$y(x_2)$
0.1	2.98×10^{-4}	8.27×10^{-2}	1.38×10^{0}	3.20×10^{0}
0.2	3.17×10^{-6}	5.64×10^{-4}	9.02×10^{-1}	7.36×10^{-1}
0.3	2.06×10^{-5}	1.27×10^{-4}	1.09×10^{0}	2.58×10^{0}
0.4	8.60×10^{-8}	8.87×10^{-7}	9.09×10^{-1}	5.32×10^{0}
0.5	2.20×10^{-8}	2.40×10^{-8}	8.84×10^{-1}	2.10×10^{0}
0.6	2.50×10^{-8}	1.01×10^{-7}	7.22×10^{-1}	1.51×10^{0}
0.7	4.86×10^{-8}	4.20×10^{-9}	7.15×10^{-1}	1.71×10^{0}
0.8	1.95×10^{-8}	1.67×10^{-8}	6.42×10^{-1}	2.57×10^{0}
0.9	2.19×10^{-8}	7.43×10^{-8}	5.78×10^{-1}	1.39×10^{0}
1.0	3.68×10^{-8}	0.00×10^{0}	5.68×10^{-1}	1.67×10^{-1}

V. CONCLUSION

An A-stable backward difference second derivative linear multistep method for solving stiff ordinary differential equation via multi-step interpolation and collocation methods has be studied in this research. In deriving the methods, we use the power series as a basis function to obtain the main continuous and discrete schemes following the method of (Mohammed and Raphael, 2015), (Nwachukwu and Okor, 2018), (Skwame, Sunday and Sabo, 2018), (Abdelrahhim and Omar, 2016) and currently (Cash, 1981, 1983) and (Mohammed and Omar, 2017). The analysis of the basic properties of the new methods which include the order, error constant, consistency, zero-stability and stability regions has been studied in this research, and it was found to be consistent, zero-stable, with uniformly order twelve. We further plotted the region of absolute stability of our methods which the method is A-stable. The newly constructed methods was applied to solve the systems stiff initial value problems of general of second-order ordinary differential equations and the results was compared with the existing methods of (Skwame, Kumleng and Bakari, 2017), (Althemai, Skwame and Donald, 2014). It is evident that, our method performs better than the existing methods.

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